Instructions.

- Use PENCIL AND ERASER.
- You must not communicate with other students during this test.
- No written materials of any kind allowed.
- No phones, calculators, iPods or electronic devices of any kind are allowed for ANY reason, including checking the time (you may use a simple wristwatch).
- Here's how it will be graded:

  1. Any substantial effort gets 7/10.
  2. 3 correct answers gets 8/10
  3. 4 correct answers gets 9/10.
  4. 5 correct answers gets 10/10
  5. Question 4 will be counted as 2 problems. So there are 6 problems.

Violations of academic integrity (in other words, cheating) will be taken extremely seriously, and will be handled under the procedures of Article I, Part 4 of the student code.
Q1 ] Determine the general solution of

\[ xy' + x^2 y = x^2 \]

\[ x \frac{dy}{dx} + x^2 y = x = 1 \quad \frac{dy}{dx} + xy = x \]

\[ -1 \frac{dy}{dx} = x(1-y) \quad \text{(Separable)} \]

\[ \Rightarrow - \frac{dy}{1-y} = x \, dx \implies \ln |1-y| = \frac{x^2}{2} + C' \]

\[ \Rightarrow \ln |1-y| = - \frac{x^2}{2} + C'' \]

\[ \Rightarrow 1 - y = e^{- \frac{x^2}{2}} \cdot e^C \implies y = 1 - e^{- \frac{x^2}{2}} \cdot C \]

\[ \Rightarrow y = 1 - e^{- \frac{x^2}{2}} \cdot C \]
Q2] A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

\[
\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (ft/s^2)
\]

If the car starts from rest \((x_0 = 0, \ v_0 = 0)\) find the distance it has traveled at the end of the first 10 s and its velocity at that time.

If \(v_0\) is given, since \(\frac{dx}{dt} = v\), we can integrate.

So, find \(x\):

\[
\int \frac{dx}{dt} = (0.12) \int t^2 dt + (0.6) \int t dt
\]

\[
= 0.4t^3 + 0.3t^2 + C_1
\]

\(v(0) = 0 \Rightarrow C_1 = 0\)

\(v(t) = (0.12) \frac{t^3}{3} + (0.6) \frac{t^2}{2}\)

Now, \(\frac{dx}{dt} = v\), so:

\[
x(t) = \int dx = (0.12) \frac{t^4}{12} + (0.6) \frac{t^3}{6} + C_2
\]

\(x(0) = 0 \Rightarrow C_2 = 0\)

\(\therefore x(t) = (0.12) \frac{t^4}{12} + (0.6) \frac{t^3}{6}\)

and

\[
x(10) = 0.12 \frac{(10)^4}{12} + (0.6) \frac{(10)^3}{6}\]
Q3] An object is moving along an axis with the acceleration given as

\[ a(t) = 2 + 2\pi \cos(\pi t) \]

where \( t \) denotes the time. Assume that the initial velocity is \( v(0) = 0 \). What is the net change of position between time \( t = 0 \) and \( t = 1 \)?

\[\begin{align*}
\text{Find } & \quad x(1) - x(0) \\
\text{where } & \quad v(t) = \int [2 + 2\pi \cos(\pi t)] \, dt = 2t + \frac{2\pi}{\pi} \sin(\pi t) + C_1 \\
& \quad v(0) = 0 \Rightarrow C_1 = 2 \\
& \Rightarrow v(t) = 2t + 2\sin(\pi t) \\
\text{Now find } & \quad x(t) = \int v(t) \, dt = \int (2t + 2\sin(\pi t)) \, dt \\
& \quad = t^2 - \frac{2}{\pi} \cos(\pi t) + C_2 \\
\end{align*}\]

\[\begin{align*}
x(1) - x(0) & = (1 - \frac{2}{\pi} \cos(\pi) + C_2) - (\frac{2}{\pi} \cos(0) + C_2) \\
& = 1 - \frac{2}{\pi} + C_2 - \frac{2}{\pi} - C_2 \\
& = 1
\end{align*}\]
Q4] Find the general solution for the following differential equations.

1. $yy' + x = \sqrt{x^2 + y^2}$

\[
\frac{dy}{dx} = -\frac{x}{y} + \sqrt{\frac{x^2 + y^2}{y^2}}
\]

Let $u = \frac{y}{x}$, i.e. $ux = y$.

Then $u'x + u = y'$.

So:

\[
\frac{du}{dx}.x + u = \frac{dy}{dx}
\]

New DE: $u + x \frac{du}{dx} = -\frac{1}{u} + \frac{\sqrt{1 + \frac{1}{u^2}}}{u}$

\[
x \frac{du}{dx} = \frac{\sqrt{u^2 + 1} - (1 + u^2)}{u} \Rightarrow \int \frac{(du).u}{(1 + u^2) + \sqrt{u^2 + 1}} = \int \frac{dx}{x}
\]

$t = 1 + \frac{1}{u^2} \Rightarrow 2udu = dt$

$\int \frac{1}{\sqrt{t}(1 - \sqrt{t})} = \ln x + C$

$\frac{1}{2} \int \frac{dt}{\sqrt{t}} = \ln x + C$.

\[
-\frac{1}{u} \Rightarrow -\frac{1}{u} du = \ln x + C
\]

\[
-\ln |u| = \ln x + C
\]

\[
-\ln |1 - \sqrt{1 + u^2}| = \ln x + C
\]

\[
-\ln |1 - \sqrt{1 + \frac{1}{u^2}}| = \ln x + C
\]

\[
\text{Exp.}\] \quad \ln x + \ln |1 - \sqrt{1 + u^2}| \quad \Rightarrow x(1 - \sqrt{1 + u^2})C = 1
\]

\[
\Rightarrow x(1 - \sqrt{1 + \frac{y^2}{x^2}}) = C
\]
Q5] Find the general solution of the given differential equation:

\[
\frac{dy}{dx} = \frac{\sqrt{y} - y}{\tan x}
\]

\[
\int \frac{dy}{\sqrt{y}(1-\sqrt{y})} = \int \frac{dx}{\sin x \cos x}
\]

On the left, let \(1-\sqrt{y} = u\)

\[
\Rightarrow \frac{1}{2\sqrt{y}} dy = du.
\]

On the right, let \(u = \sin x\)

\[
2 \ln|u| = \ln u + \ln C
\]

Then, \(\cos x dx = du\)

Take exp sides of \(u^2 = \sqrt{C}\)

\[
|u| = \sqrt{C}
\]

\[
(1-\sqrt{y})^2 = (\sin x)^2 . C
\]
\[ xy' - y = \sqrt{x^2 + y^2} \]

\[ x \frac{dy}{dx} - y = \sqrt{x^2 + y^2} \]

\[ \Rightarrow \frac{dy}{dx} - \frac{y}{x} = \sqrt{\frac{x^2 + y^2}{x^2}} \]

Let \( u = \frac{y}{x} \).

Then, \( u + x \frac{du}{dx} = \sqrt{1 + u^2} \) is the new DE.

\( u + x \frac{du}{dx} = \frac{1}{x} \) (up to here get full credit)

\( \Rightarrow \frac{du}{\sqrt{1 + u^2}} = \frac{dx}{x} \)

Substitution on the left: \( u = \tan \theta \).

Then, \( du = \sec^2 \theta \, d\theta \)

\[ \int \frac{dx}{\sqrt{1 + \tan^2 \theta}} = \ln |x| + C' \]

\[ \int \sec \theta \, d\theta = \ln |x| + C' \]

\[ \ln (u + \sqrt{u^2 + 1}) = \ln |x| + C' \]

\[ \Rightarrow y + \sqrt{y^2 + x^2} = Cx^2 \]