ON THE DISTRIBUTION OF ANGLES BETWEEN GEODESIC RAYS
ASSOCIATED WITH HYPERBOLIC LATTICE POINTS

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Abstract. For every two points \( z_0, z_1 \) in the upper-half plane \( \mathbb{H} \), consider all elements \( \gamma \) in the principal congruence group \( \Gamma(N) \), acting on \( \mathbb{H} \) by fractional linear transformations, such that the hyperbolic distance between \( z_1 \) and \( \gamma z_0 \) is at most \( R > 0 \). We study the distribution of angles between the geodesic rays \( [z_1, \gamma z_0] \) as \( R \to \infty \), proving that the limiting distribution exists independently of \( N \) and explicitly computing it. When \( z_1 = z_0 \) this is found to be the uniform distribution on the interval \( [-\frac{\pi}{2}, \frac{\pi}{2}] \).

1. Introduction

In this paper the group \( SL_2(\mathbb{R}) \) acts on the upper half-plane \( \mathbb{H} \) by linear fractional transformations \( z \mapsto g z = \frac{az + b}{cz + d}, \quad g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{R}), \quad z \in \mathbb{H} \). The hyperbolic ball \( B(z_0, R) = \{ z \in \mathbb{H} : g(z_0, z) \leq R \} \) of center \( z_0 = x_0 + iy_0 \in \mathbb{H} \) and radius \( R \) coincides with the Euclidean ball of center \( x_0 + iy_0 \) of radius \( y_0 \sinh R \) and radius \( y_0 \sinh R \approx \frac{1}{2} y_0 e^R \). Let \( \Gamma \) be a discrete subgroup of \( SL_2(\mathbb{R}) \). The hyperbolic circle problem of estimating for fixed \( z_0, z_1 \in \mathbb{H} \) and \( R \to \infty \) the cardinality of the set \( \Gamma_{z_0, R} = \{ \gamma \in \Gamma : \gamma z_0 \in B(z_0, R) \} \), or slightly more generally of \( \{ \gamma \in \Gamma : \gamma(z_0, z_1) \leq R \} \), has been thoroughly studied with various methods (see, e.g., [4, 6, 7, 8, 9, 10, 14], and [10, 11, 12, 13, 17] for some higher dimensional analogs of the problem).

We consider another natural problem concerning the distribution of hyperbolic lattice points in angular sectors. For \( z_0, z_1 \in \mathbb{H} \) and \( g \in SL_2(\mathbb{R}) \), let \( \theta_{z_0, z_1}(g) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) denote the angle between the geodesic ray \( [z_1, g z_0] \) and the vertical geodesic \( [z_1, \infty] \). Given a compact set \( \Omega \subset \mathbb{H} \) and a number \( \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), the proportion of points in the \( \Gamma \)-orbit of \( z_0 \) inside \( \Omega \) such that \( \theta_{z_0, z_1}(\gamma) \leq \omega \) is given by

\[
P_{\Gamma, \Omega, z_0, z_1}(\omega) = \frac{\# \{ \gamma \in \Gamma : \gamma z_0 \in \Omega, \theta_{z_0, z_1}(\gamma) \leq \omega \}}{\# \{ \gamma \in \Gamma : \gamma z_0 \in \Omega \}}.
\]

It is natural to investigate the existence of the limiting distribution

\[
P_{\Gamma, z_0, z_1}(\omega) = \lim_{R \to \infty} P_{\Gamma, B(z_0, R), z_0, z_1}(\omega) = \lim_{R \to \infty} \frac{\# \{ \gamma \in \Gamma_{z_0, R} : \theta_{z_0, z_1}(\gamma) \leq \omega \}}{\# \Gamma_{z_0, R}}.
\]

In this paper we consider the case where

\[
\Gamma = \Gamma(N) = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) : a, d \equiv 1, \quad b, c \equiv 0 \pmod{N} \right\}
\]

is the principal congruence subgroup of level \( N \), which is the kernel of the natural surjective morphism \( SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N) \). This is a normal subgroup of \( \Gamma(1) = SL_2(\mathbb{Z}) \) of index

\[
(1.1) \quad [\Gamma(1) : \Gamma(N)] = N^3 \prod_{p \mid N} (1 - p^{-2}).
\]

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For every $g = (A \, B) \in SL_2(\mathbb{R})$ the hyperbolic distance $d(i, gi)$ is given by
\begin{equation}
\cosh d(i, gi) = 1 + \frac{|i - gi|^2}{2 \text{Im}(gi)} = \frac{A^2 + B^2 + C^2 + D^2}{2}.
\end{equation}

Denote
\begin{equation}
C_N = \sum_{n \geq 1} \mu(n) \frac{n^2}{n^2} = \prod_{p \mid N} (1 - p^{-2}) = \frac{1}{\zeta(2)} \prod_{p \mid N} (1 - p^{-2})^{-1}.
\end{equation}

For every $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1 \in \mathbb{H}$, denote $x_\ast = \frac{x_1 - x_0}{y_0}, y_\ast = \frac{y_1}{y_0}$, and consider the continuous function $\Xi_{x_\ast, y_\ast}$ on $[-\pi, \pi]$ defined by
\begin{equation}
\Xi_{x_\ast, y_\ast}(\omega) = \frac{1}{\pi} \arctan \left( x_\ast + y_\ast \tan \frac{\omega}{2} \right) + \frac{1}{\pi} \arctan \left( x_\ast - y_\ast \cot \frac{\omega}{2} \right)
\end{equation}
\begin{equation}
- \frac{1}{\pi} \arctan(x_\ast + y_\ast) - \frac{1}{\pi} \arctan(x_\ast - y_\ast) + \begin{cases} 1 & \text{if } \omega > 0, \\ 0 & \text{if } \omega < 0. \end{cases}
\end{equation}

The main result of this paper is

**Theorem 1.** For every positive integer $N$ and $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1 \in \mathbb{H}$, as $R \to \infty$,
\begin{equation}
\# \{ \gamma \in \Gamma(N) : -\pi \leq \theta_{z_0, z_1}(\gamma) \leq \omega \} \sim \frac{\pi^2 C_N \Xi_{x_\ast, y_\ast}(\omega)}{N^3} e^R + O_{\varepsilon,N,z_0,z_1} \left( e^{(7/8+\varepsilon)R} \right).
\end{equation}

In particular the limiting distribution $\mathbb{P}_{\Gamma(N), z_0, z_1}$ exists and is given by
\[
\mathbb{P}_{\Gamma(N), z_0, z_1}(\omega) = \frac{1}{\pi} \int_{-\pi/2}^{\omega} \varrho_{z_0, z_1}(t) \, dt, \quad \omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],
\]
where
\[
\varrho_{z_0, z_1}(t) = \frac{2y_0 y_1 (y_0^2 + (x_1 - x_0)^2 + y_1^2)}{(y_0^2 + (x_1 - x_0)^2 + y_1^2)^2 - ((y_0^2 + (x_1 - x_0)^2 - y_1^2) \cos t + 2(x_1 - x_0) y_1 \sin t)^2}.
\]

Taking $z_1 = z_0$ we infer

**Corollary 1.** The angles $\theta_{z_0, z_0}(\gamma), \gamma \in \Gamma(N) z_0 R$, are uniformly distributed as $R \to \infty$.

The converse is also seen to be true, so that the angles $\theta_{z_0, z_1}(\gamma)$ are uniformly distributed as $R \to \infty$ if and only if $z_1 = z_0$. In the Euclidean situation these angles are uniformly distributed regardless of the choice of $z_1$ and $z_0$.

Our method of proof is number theoretical and relies on the Weil bound for Kloosterman sums [18], as previously used (for instance) in [1, 2, 3, 5, 8]. In the process we also derive, as a consequence of the proof of Theorem 1, an asymptotic formula for the number of hyperbolic lattice points in large balls.

**Corollary 2.** For every positive integer $N$ and every $z_0 \in \mathbb{H}$, as $R \to \infty$,
\begin{equation}
\# \Gamma(N) z_0 R = \frac{6e^R}{\Gamma(1) : \Gamma(N)} + O_{\varepsilon,N,z_0} \left( e^{(7/8+\varepsilon)R} \right).
\end{equation}

Denoting by $\mu$ the hyperbolic area in $\mathbb{H}$, the main term in (1.5) is $\sim \frac{2\mu(B(z_0,R))}{\mu(\Gamma(N)z_0)}$ as $R \to \infty$.

For $N = 1$ formula (1.6) has been proved using Kloosterman sum estimates in [8]. Better error terms with exponent as low as $\frac{1}{2}$ can be obtained using Selberg’s theory on the spectral decomposition of $L^2(\Gamma(N) \backslash \mathbb{H})$ (see [14] for exponent $\frac{1}{3}$ and [9] for exponent $\frac{1}{2}$) and lower bounds for the first eigenvalue of the Laplacian on $\Gamma(N) \backslash \mathbb{H}$ (see [15], [9], and [16] for a review of recent developments). Similar results hold when $\Gamma(N)$ is replaced by any of the congruence groups.
\( \Gamma_0(N) = \{ \gamma \in \Gamma(1) : c \equiv 0 (\text{mod } N) \} \) or \( \Gamma_1(N) = \{ \gamma \in \Gamma(1) : a, d \equiv 1, c \equiv 0 (\text{mod } N) \} \), or when \( \Gamma(N) \to R \) is replaced by \( \{ \gamma \in \Gamma(N) : \varrho(\gamma z_0, z_1) \leq R \} \) for fixed \( z_0, z_1 \in \mathbb{H} \).

There are two natural problems that arise in this context. It would be interesting to know how large is the class of discrete subgroups of \( SL_2(\mathbb{R}) \) for which the analogue of Theorem 1 holds. It would also be interesting to study the spacing statistics (both consecutive spacings and correlations) of these angles when \( z_0 = z_1 \).

2. Reducing the problem to a counting problem

Given \( z_0 = x_0 + iy_0 \in \mathbb{H} \) and \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma(1) \), consider

\[
g_0 = \left( \frac{\sqrt{y_0} + x_0}{\sqrt{y_0}} \right), \quad g = g_0^{-1} \gamma g_0 = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in SL_2(\mathbb{R}),
\]

with

\[
A = a - cx_0, \quad B = \frac{(a - cx_0)x_0 + b - dx_0}{y_0}, \quad C = cy_0, \quad D = cx_0 + d.
\]

Since \( g_0i = z_0 \) we have

\[
cosh \varrho(z_0, \gamma z_0) = \cosh \varrho(g_0i, g_0gi) = \cosh \varrho(i, gi) = \frac{A^2 + B^2 + C^2 + D^2}{2}.
\]

Take \( Q^2 = 2 \cosh R \sim e^R \). As a result of (2.2) we are interested in those \( \gamma \in \Gamma(N) \) for which \( A^2 + B^2 + C^2 + D^2 \leq Q^2 \). The only matrices \( \gamma \in \Gamma(1) \) with \( c = 0 \) are \( \pm I \) and as a result we can assume next that \( C \neq 0 \). We will also assume that \( A \neq 0 \).

The geodesic joining the points \( z_* = x_* + iy_* \in \mathbb{H} \) and \( gi, g = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in SL_2(\mathbb{R}) \), is the half-circle of center \( \alpha \) and radius \( r \), where

\[
|\alpha - z_*| = |\alpha - gi| = r.
\]

This gives

\[
|\alpha - x_* - iy_*|^2 = \left| \alpha - \frac{iA + B}{iC + D} \right|^2 = \frac{|i(C\alpha - A) + D\alpha - B|^2}{|iC + D|^2},
\]

and after cancelling out the terms containing \( \alpha^2 \) we obtain

\[
2\alpha(x_*E - F) = (x_*^2 + y_*^2)E - G,
\]

with

\[
E = C^2 + D^2, \quad F = AC + BD, \quad G = A^2 + B^2,
\]

leading to

\[
\tan \theta_{i,z_*}(g) = \frac{y_*}{x_* - \alpha} = \frac{2y_*(F - x_*E)}{(y_*^2 - x_*^2)E + 2x_*F - G}.
\]

We will keep \( z_0 \) and \( z_1 \) fixed throughout. Taking

\[
z_* = g_0^{-1}z_1 = \frac{x_1 - x_0 + iy_1}{y_0}
\]

we have \( g_0(x_* + it) = x_1 + iy_0t, \ t > 0 \), so that

\[
\theta_{z_0,z_1}(\gamma) = \zeta(x_1 + i\infty, z_1, \gamma z_0) = \zeta[g_0(x_* + i\infty), g_0z_*] = \zeta[x_* + i\infty, z_*, gi] = \theta_{i,z_*}(g),
\]

and therefore

\[
(2.3) \quad \tan \theta_{z_0,z_1}(\gamma) = \frac{y_*}{x_* - \alpha} = \frac{2y_*(F - x_*E)}{(y_*^2 - x_*^2)E + 2x_*F - G}.
\]
When $|A| \leq |D|$ we use
\[
\begin{align*}
\left| \frac{F - A}{G} \right| &= \frac{|D|}{|C||C^2 + D^2|} \leq \frac{1}{2C^2}, \\
\left| \frac{G}{E} \frac{A^2}{C^2} \right| &= \frac{|BC + AD|}{C^2(C^2 + D^2)} = \frac{|2AD - 1|}{C^2(C^2 + D^2)} \leq \frac{2}{C^2} + \frac{1}{C^4},
\end{align*}
\]
to derive
\[
\tan \theta_{z_0, z_1}(\gamma) = \frac{2y_*(\frac{E}{C} - x_*)}{y_*^2 - x_*^2 + 2x_* \frac{E}{C} - \frac{E}{E}} = \frac{2y_*(\frac{A}{C} - x_*)}{y_*^2 - (\frac{A}{C} - x_*)^2 + O_{z_*}(\frac{1}{C^2} + \frac{1}{C^4})},
\]
When $|D| \leq |A|$ we use
\[
\begin{align*}
\left| \frac{F}{G} - \frac{C}{A} \right| &= \frac{|B|}{|A|(A^2 + B^2)} \leq \frac{1}{2A^2}, \\
\left| \frac{E}{G} - \frac{C^2}{A^2} \right| &= \frac{|2AD - 1|}{A^2(A^2 + B^2)} \leq \frac{2}{A^2} + \frac{1}{A^4},
\end{align*}
\]
to derive
\[
\tan \theta_{z_0, z_1}(\gamma) = \frac{2y_*(\frac{E}{C} - x_*)}{(y_*^2 - x_*^2) \frac{E}{C} + 2x_* \frac{E}{C} - 1} = \frac{2y_*(\frac{C}{A} - x_* \frac{C}{A^2})}{2x_* \frac{C}{A} - 1 + O_{z_*}(\frac{1}{A^2} + \frac{1}{A^4})}
\]
\[
\leq \frac{2y_*(\frac{A}{C} - x_*)}{y_*^2 - (\frac{A}{C} - x_*)^2 + O_{z_*}(\frac{1}{C^2} + \frac{1}{C^4})}.
\]
For $\lambda > 0$ set
\[
-\alpha_1 := -1 - \frac{\sqrt{1 + \lambda^2}}{\lambda} < -1 < \alpha_2 := -1 + \frac{\sqrt{1 + \lambda^2}}{\lambda} = \frac{1}{\alpha_1} < 1.
\]
For $\lambda < 0$ set
\[
-1 < \alpha_1^* := 1 - \frac{\sqrt{1 + \lambda^2}}{|\lambda|} < 0 < \alpha_2^* := 1 + \frac{\sqrt{1 + \lambda^2}}{|\lambda|} = -\frac{1}{\alpha_1^*}.
\]
Letting $\lambda = \tan \omega, \omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\alpha_1 = \cot \frac{\omega}{2}, \alpha_2 = \tan \frac{\omega}{2}$ for $\omega > 0$, and $\alpha_1^* = \tan \frac{\omega}{2}, \alpha_2^* = -\cot \frac{\omega}{2}$ for $\omega < 0$. A plain calculation gives
\[
\frac{2y_*(X - x_*)}{y_*^2 - (X - x_*)^2} < \lambda \iff X - x_* \in \mathcal{S}(y_*, \lambda),
\]
with
\[
\mathcal{S}(y_*, \lambda) = \begin{cases} 
(-\infty, -y_\alpha \alpha_1) \cup (-y_\alpha, y_\alpha \alpha_2) \cup (y_\alpha, \infty) & \text{if } \lambda > 0, \\
(-y_\alpha, 0) \cup (y_\alpha, \infty) & \text{if } \lambda = 0, \\
(-y_\alpha, y_\alpha \alpha_1^*) \cup (y_\alpha, y_\alpha \alpha_2^*) & \text{if } \lambda < 0.
\end{cases}
\]
For fixed $\lambda \in \mathbb{R}, z_* \in \mathbb{H}$, and $|\varepsilon_1|, |\varepsilon_2|$ small, the roots $X_{\pm}(\varepsilon_2)$ of $y_*^2 - (X - x_*)^2 + \varepsilon_2 = 0$ and $\bar{X}_{\pm}(\varepsilon_1, \varepsilon_2)$ of $2y_*(X - x_*) + \varepsilon_1 - \lambda(y_*^2 - (X - x_*)^2 + \varepsilon_2) = 0$ satisfy
\[
|X_{\pm}(\varepsilon_2) - X_{\pm}(0)| = \sqrt{y_*^2 + \varepsilon_2} - y_* \leq \frac{|\varepsilon_2|}{y_*},
\]
and respectively
\[
|\bar{X}_{\pm}(\varepsilon_1, \varepsilon_2) - \bar{X}_{\pm}(0, 0)| = \frac{|\varepsilon_1 - \lambda \varepsilon_2|}{y_* \sqrt{1 + \lambda^2} + \sqrt{y_*^2(1 + \lambda^2) - \lambda^2 \varepsilon_2} - \lambda^2 \varepsilon_2} \leq \frac{|\varepsilon_1 - \lambda \varepsilon_2|}{y_* \sqrt{1 + \lambda^2}} \leq \frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{y_*}.\]
In conjunction with (2.4)–(2.7) this shows, in both cases \(|A| \leq |D|\) and \(|D| \leq |A|\), that there is a constant \(K_1 = K_1(z_*) > 0\) such that, for any \(\gamma \in \Gamma(N)\),

\[
\tan \theta_{z_0,z_1}(\gamma) \leq \lambda \implies \frac{A}{C} \in x_\ast + \mathfrak{G}(y_\ast, \lambda) + \left[ -K_1 H(\gamma), K_1 H(\gamma) \right],
\]

where \(H(\gamma) = \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{c^2 + d^2}\).

We wish to discard those \(\gamma\) for which one of \(|A|\), \(|B|\), \(|C|\), \(|D|\) is small. Note first that, as a result of (2.1), there is a constant \(K_0 = K_0(z_0)\) such that \(a^2 + b^2 + c^2 + d^2 \leq K_0Q^2\) whenever \(A^2 + B^2 + C^2 + D^2 \leq Q^2\). For every \(K > 0\) let \(\mathcal{E}_A(K) = \mathcal{E}_{A,Q,z_0}(K)\) denote the number of \(\gamma \in \Gamma(1)\) for which \(A^2 + B^2 + C^2 + D^2 \leq Q^2\) and \(|A| = |a - c x_0| \leq K\). Define similarly \(\mathcal{E}_B(K), \mathcal{E}_C(K), \mathcal{E}_D(K)\).

**Lemma 1.** (i) For every \(z_0 \in \mathbb{H}\) and \(K \geq 1\)

\[
\max \{ \mathcal{E}_A(K), \mathcal{E}_C(K), \mathcal{E}_D(K) \} \ll_{z_0} KQ \log Q \quad (Q \to \infty).
\]

(ii) For every \(z_0 \in \mathbb{H}\) and \(\alpha \in (0,1)\)

\[
\mathcal{E}_B(Q^\alpha) \ll_{z_0} Q^{(3+\alpha)/2} \log Q \quad (Q \to \infty).
\]

**Proof.** (i) The congruence \(bc = 1 \pmod{|a|}\) shows, for fixed \(c\) and \(a \neq 0\), that the integer \(b\) is uniquely determined \((\mod{|a|})\), so it takes \(\ll \frac{Q}{|a|} \) values. This gives

\[
\mathcal{E}_C(K) \ll 2 + \left(\frac{2K}{y_0} + 1\right) \sum_{1 \leq |a| \leq KQ} \frac{Q}{|a|} \ll_{z_0} KQ \log Q.
\]

To prove \(\mathcal{E}_A(K) \ll_{z_0} KQ \log Q\) note that, for fixed \(c \in [-K_0Q, K_0Q]\), there are at most \(2K + 1\) integers \(a\) such that \(|a - c x_0| \leq K\). For each such \(a\), the congruence \(ad = 1 \pmod{|c|}\) uniquely determines \(d\) \((\mod{|c|})\), so the number of admissible triples \((a,d,b)\) is \(\ll \frac{KQ}{|c|}\), and summing over \(c\) we find as above \(\mathcal{E}_A(K) \ll_{z_0} KQ \log Q\). The proof of \(\mathcal{E}_D(K) \ll_{z_0} KQ \log Q\) is similar.

(ii) Let \(\mathcal{E}_A(K)^c\), respectively \(\mathcal{E}_D(K)^c\), denote the complement of \(\mathcal{E}_A(K)\), respectively \(\mathcal{E}_D(K)\), in \(\{\gamma \in \Gamma(1) : A^2 + B^2 + C^2 + D^2 \leq Q^2\}\). Write \(\alpha = 2\alpha' - 1\), \(\frac{1}{2} < \alpha' < 1\), so that \(1 + \alpha' = \frac{3+\alpha}{2}\).

For every \(\gamma \in \mathcal{E}_A(Q^{\alpha'} + 1)^c \cap \mathcal{E}_D(Q^{\alpha'} + 1)^c\) we have

\[
|B| = \frac{|AD - 1|}{|C|} > \frac{Q^{2\alpha'}}{Q} = Q^\alpha,
\]

showing that \(\mathcal{E}_B(Q^\alpha) \ll \mathcal{E}_A(Q^{\alpha'} + 1) \cup \mathcal{E}_D(Q^{\alpha'} + 1)\), and so \(\mathcal{E}_D(Q^\alpha) \ll_{z_0} Q^{1+\alpha'} \log Q\). \(\square\)

Note also that

\[
\left| (A^2 + B^2 + C^2 + D^2) - (C^2 + A^2) \left(1 + \frac{D^2}{C^2}\right) \right| = \left| \frac{AD + BC}{C^2} \right| = \frac{2BC + 1}{C^2}
\]

\[
\leq \frac{2|B|}{|C|} + \frac{1}{C^2} \ll_{z_0} \frac{Q}{|c|} + \frac{1}{c^2} \ll Q.
\]

The relations (2.8) and (2.9) lead us to estimate the number

\[
\mathfrak{N}_Q(N, z_0; \beta) := \# \{ \gamma \in \Gamma(N) : \frac{A}{C} \leq \beta, (C^2 + A^2) \left(1 + \frac{D^2}{C^2}\right) \leq Q^2 \} \quad (Q \to \infty).
\]
3. SOME COUNTING IN Γ(N)

In this section we prove some counting results which will be further used in the proof of
Theorem 1 in the next section. Let c and N ≥ 1 be integers and consider the sum

\[ \Phi_N(c) := \sum_{n|c \text{ and } (n,N)=1} \frac{\mu(n)}{n}. \]

We first estimate the number

\[ N_{c,N}(I_1 \times I_2) := \# \{(a,d) \in I_1 \times I_2 : a \equiv 1, \ d \equiv 1 \pmod{N}, \ ad \equiv 1 \pmod{Nc} \}, \]

with fixed N and c, and with a and d in prescribed (short) intervals. The next result extends
Lemma 1.6 in [2] from Γ(1) to Γ(N).

Proposition 1. For a fixed positive integer N and intervals I_1, I_2 of length less than |c|

\[ N_{c,N}(I_1 \times I_2) = \frac{\Phi_N(c)}{|N|^2} |I_1| |I_2| + O_{c,N}(|c|^{1/2+\varepsilon}) \quad (|c| \to \infty). \]

Proof. Replacing (b, c) by (−b, −c) we can assume c > 0. In this case we write

\[ N_{c,N}(I_1 \times I_2) = \frac{1}{Nc} \sum_{\substack{x \in I_1 \\ (x,Nc)=1}} \sum_{\substack{y \in I_2 \\ y \equiv 1 \pmod{N}}} \sum_{k \pmod{Nc}} \sum_{\ell \equiv k \pmod{Nc}} e \left( \frac{k(y - \bar{x})}{Nc} \right) = M + E, \]

where \( \bar{x} \) is the multiplicative inverse of \( x \pmod{Nc} \) and \( e(t) = \exp(2\pi it) \). The contribution

\[ (3.1) \quad M = \frac{1}{Nc} \sum_{0 \leq k < Nc \atop c|k} \sum_{\substack{y \in I_2 \\ y \equiv 1 \pmod{N}}} \sum_{\ell \equiv k \pmod{Nc}} e \left( \frac{\ell(y - \bar{x})}{N} \right) \]

of terms with \( c \mid k \) to \( N_{c,N}(I_1, I_2) \) will be treated as a main term, while the contribution

\[ (3.2) \quad E = \frac{1}{Nc} \sum_{0 \leq k < Nc \atop c|k} \sum_{\substack{x \in I_1 \\ (x,Nc)=1}} \sum_{\ell \equiv k \pmod{Nc}} e \left( \frac{\ell y}{Nc} \right) \sum_{\substack{x \in I_1 \\ (x,Nc)=1}} e \left( -\frac{k\bar{x}}{Nc} \right) \]

of terms with \( c \nmid k \) will be treated as an error term.

To estimate E consider for I interval and \( q \in \mathbb{N}, \ m,n \in \mathbb{Z} \), the incomplete Kloosterman sum

\[ S_I(m, n; q) := \sum_{a \equiv 1 \atop (a,q)=1} \frac{e \left( ma + na \bar{a} \right)}{q}, \]

where \( \bar{a} \) is the multiplicative inverse of \( a \pmod{q} \). The complete Kloosterman sum \( S(m, n; q) \)

is just \( S_{[0,q-1]}(m, n; q) \). For any interval \( I \subseteq [0, q-1] \) and integers \( m, n \), not both divisible by \( q \), the Weil bound on Kloosterman sums leads (cf., e.g., [2, Lemma 1.6]) to

\[ (3.3) \quad |S_I(m, n; q)| \ll_{\varepsilon} (n, q)^{1/2} q^{1/2+\varepsilon}. \]

Writing now the inner sum in (3.2) as

\[ \sum_{x \in I_1 \atop (x,Nc)=1} e \left( -\frac{k\bar{x}}{Nc} \right) \frac{1}{N} \sum_{s \pmod{N}} e \left( \frac{s(x - 1)}{N} \right) = \frac{1}{N} \sum_{s \pmod{N}} e \left( -\frac{s}{N} \right) S_{I_1}(cs, -k; Nc) \]
and applying (3.3) we find

\[
|E| \ll \varepsilon \left( \frac{(NC)^{1/2+\varepsilon}}{NC} \sum_{0 \leq k < NC} (k, NC)^{1/2} \right) \sum_{y \equiv 1 \pmod{N}} \sum_{y \in I_2} e^{\left( \frac{ky}{NC} \right)}.
\]

Treating the inner sum above as a geometric progression of ratio \(e^{|\varepsilon|} \) and using the inequality \(|\sin \pi t| \geq 2|t| = 2 \text{dist}(t,Z)\), \(t \in \mathbb{R}\), the inner sum above is \(\leq \min \{ |I_2|, \frac{1}{2|k/|c||} \} \). Employing also the inequality \((k, NC) \leq (k, c)N\) we further find

\[
|E| \ll \varepsilon N^{1+\varepsilon} \left( \frac{c^{1/2+\varepsilon}}{c} \right) \sum_{0 < \ell < c} \left( \frac{\ell, c}{2|\varepsilon|} \right) \ll N^{1+\varepsilon} \varepsilon^{-1/2+\varepsilon} \sum_{d|c} \sum_{m \leq \bar{N}} d^{1/2} \frac{c}{dm}
\]

\[
\leq N^{1+\varepsilon} c^{1/2+\varepsilon} \sum_{d|c} d^{-1/2} \log c \ll \varepsilon N c^{1/2+2\varepsilon}.
\]

Concerning the main term \(M\), from \(x\bar{x} = 1 (\pmod{N})\) and \(x = 1 (\pmod{N})\) we infer \(\bar{x} = 1 (\pmod{N})\). The inner sum in (3.1) is equal to \(N\) and we get

\[
M = \frac{1}{c} \sum_{x \in I_1} \sum_{d|c} \sum_{y \equiv 1 \pmod{N}} 1 = \frac{1}{c} \left( \frac{|I_2|}{N} + O(1) \right) \sum_{x \in I_1} \sum_{d|c} \sum_{y \equiv 1 \pmod{N}} 1.
\]

Using \(x = 1 (\pmod{N})\) and Möbius summation, the latter sum above can also be expressed as

\[
\sum_{x \equiv 1 \pmod{N}} \sum_{d|x} \mu(d) = \sum_{x \equiv 1 \pmod{N}} \sum_{d|x} \mu(d) = \sum_{d|c} \sum_{x \equiv 1 \pmod{N}} 1 = \sum_{d|c} \mu(d) \left( \frac{|I_1|}{dN} + O(1) \right) = \frac{|I_1|}{N} \Phi_N(c) + O_\varepsilon(c^\varepsilon),
\]

which completes the proof. \(\square\)

**Corollary 3.** For \(I\) interval, \(C^1\) functions \(f_1, f_2 : I \to \mathbb{R}\) with \(f_1 \leq f_2\), and \(T \geq 1\) integer, the cardinality \(N_{c,N}(f_1, f_2)\) of the set

\[
\{(a, d) \in \mathbb{Z}^2 : d \in I, f_1(d) \leq a \leq f_2(d), a \equiv 1, d \equiv 1 (\pmod{N}), ad \equiv 1 (\pmod{Nc})\}
\]

can be expressed as

\[
N_{c,N}(f_1, f_2) = \Phi_N(c) \left\lfloor \frac{c}{N^2} \right\rfloor \int_I (f_2 - f_1) + \mathcal{E}_{c,N,f_1,f_2} \quad (|c| \to \infty),
\]

with

\[
\mathcal{E}_{c,N,f_1,f_2} \ll_{\varepsilon,N} \left( \frac{|I|}{|c|} \right) \left( \Phi_I(f_1) + \Phi_I(f_2) \right) + |I|^{1/2+\varepsilon} \left( 1 + \frac{|I|}{|c|} \right) \left( 1 + \frac{\|f_1\|_\infty + \|f_2\|_\infty}{|c|} \right).
\]

**Proof.** This follows from Proposition 1 as in the proof of [1, Lemma 3.1]. \(\square\)

**Lemma 2.** For every interval \(J\) and every \(C^1\) function \(f : J \to \mathbb{R}\)

\[
\sum_{c \in J} \Phi_N(c) f(c) = C_N \int_J f + O\left( \left( \|f\|_\infty + V_J(f) \right) \log \sup_{\xi \in J} |\xi| \right),
\]

with \(C_N\) as in (1.3).
Proposition 2. For every positive integer \( n \geq 1 \) consider the \( n \)-dilate function \( f_{n}(x) := f(nx), x \in [0, Q/n] \), for which \( \|f_{n}\|_{\infty} = \|f\|_{\infty}, \int_{0}^{Q/n} f_{n} = \int_{0}^{Q} f, \) and \( V_{0}^{Q/n}(f_{n}) = V_{0}^{Q}(f) \). Using Möbius and Euler-MacLaurin summation we get

\[
\sum_{c=1}^{Q} \Phi_{N}(c) f(c) = \sum_{c=1}^{Q} \sum_{n \mid c, a} \frac{\mu(n)}{n} f(c) = \sum_{n \leq Q} \frac{\mu(n)}{n} \sum_{c \leq Q/n} f(c)
\]

\[
= \sum_{n \leq Q} \frac{\mu(n)}{n} \left( \int_{0}^{Q/n} f + O(\|f\|_{\infty} + V_{0}^{Q/n}(f)) \right)
\]

\[
= \left( \sum_{n \geq 1} \frac{\mu(n)}{n^{2}} + O(\frac{1}{Q}) \right) \int_{0}^{Q} f + O\left( \log Q(\|f\|_{\infty} + V_{0}^{Q}(f)) \right)
\]

\[
= C_{N} \int_{0}^{Q} f + O\left( \log Q(\|f\|_{\infty} + V_{0}^{Q}(f)) \right),
\]

which represents the desired conclusion.

\[\Box\]

Corollary 4. For every interval \( I \) and every \( C^{1} \) function \( f : I \to \mathbb{R} \)

\[
\sum_{c \in \mathbb{Z} \setminus I} \Phi_{N}(c) f(c) = \frac{C_{N}}{N} \int_{I} f + O\left( \left( \|f\|_{\infty} + V_{I}(f) \right) \log \sup_{\xi \in I} |\xi| \right).
\]

Proof. Apply Lemma 2 to \( J = \frac{1}{N} I, f_{n}(x) = f(Nx) \), using \( \Phi_{N}(Nc') = \Phi_{N}(c') \), \( \int_{J} f_{N} = \frac{1}{N} \int_{I} f \), \( \|f_{N}\|_{\infty} = \|f\|_{\infty} \), and \( V_{I}(f) = V_{J}(f) \).

4. PROOF OF THE MAIN RESULTS

We first estimate the quantity defined in (2.10).

Proposition 2. For every positive integer \( N \) and every \( z_{0} \in \mathbb{H}, \beta \in [-\infty, \infty], \) as \( Q \to \infty \),

\[
\mathfrak{N}_{Q}(N, z_{0}; \beta) = \frac{\pi(\pi + 2 \arctan \beta)C_{N} Q^{2}}{2N^{3}} + O_{\epsilon,N,z_{0}}(Q^{7/4+\epsilon}).
\]

Proof. Define

\[
I_{c} = c x_{0} + \left\{ \begin{array}{ll}
- \sqrt{Q^{2} - c^{2} y_{0}^{2}}, & \min\{cy_{0}, \sqrt{Q^{2} - c^{2} y_{0}^{2}} \} \\
\max\{cy_{0}, -\sqrt{Q^{2} - c^{2} y_{0}^{2}} \}, & \sqrt{Q^{2} - c^{2} y_{0}^{2}} \end{array} \right\} \] 

if \( c \in [0, Q/y_{0}], \)

\[
\sqrt{\frac{Q^{2}}{c^{2} y_{0}^{2} + (a - c x_{0})^{2}} - 1},
\]

\[
f(c, a) = |c| y_{0} \sqrt{\frac{Q^{2}}{c^{2} y_{0}^{2} + (a - c x_{0})^{2}} - 1},
\]

\[
f_{1}(c, a) = -c x_{0} - f(c, a), \quad f_{2}(c, a) = -c x_{0} + f(c, a), \quad c \in [-Q/y_{0}, Q/y_{0}], \quad a \in I_{c},
\]

\[
F(c) = F_{z_{0}, \beta}(c) = \frac{2}{|c|} \int_{I_{c}} f(c, a) \, da.
\]

Writing the inequalities from (2.10) as

\[
\begin{cases}
|C| 
\leq Q, \\
-\sqrt{Q^{2} - C^{2}} \leq A \leq \sqrt{Q^{2} - C^{2}} \quad \text{and} \quad \begin{cases} A \leq \beta C \quad \text{if } C > 0, \\
A \geq \beta C \quad \text{if } C < 0,
\end{cases}
\end{cases}
\]

\[
-|C| \sqrt{\frac{Q^{2}}{C^{2} + A^{2}} - 1} \leq D \leq |C| \sqrt{\frac{Q^{2}}{C^{2} + A^{2}} - 1},
\]
and using (2.1) we gather
\[ \mathfrak{N}_Q(N, z_0; \beta) = \# \left\{ \gamma \in \Gamma(N) : |c| y_0 \leq Q, \; a \in I_c, \; d \in [f_1(c, a), f_2(c, a)] \right\} \]
\[ = \sum_{|c| \leq Q/y_0} N_{c,N}(f_1(c, \cdot), f_2(c, \cdot)). \]

(4.1)

Note that \( \max \left\{ \|f(c, \cdot)\|, V_{Lc}(f(c, \cdot)) \right\} \ll Q \) on \( I_c \), thus Corollary 3 with \( T = [Q^{1/4}] \) gives
\[ \mathfrak{N}_{c,N}(f_1(c, \cdot), f_2(c, \cdot)) = \frac{1}{N^2} \Phi_N(c) F(c) + \mathcal{E}_{c,N}, \]
with
\[ \mathcal{E}_{c,N} \ll \epsilon_{c,N} Q^{7/4} |c|^{-1} + Q^{5/4} |c|^{-1/2 + \epsilon} + Q^2 |c|^{-3/2 + \epsilon}. \]

(4.2)

Fix some constant \( \alpha \in \left[ \frac{1}{2}, \frac{3}{4} \right] \). The relation \( bc \equiv -1 \pmod{|a|} \) and the constraint \( |a| \ll z_0 Q \) give the trivial estimate
\[ \sum_{|c| \leq Q^\alpha} N_{c,N}(f_1(c, \cdot), f_2(c, \cdot)) \ll z_0 \sum_{1 \leq |a| \leq Q} Q^\alpha \frac{Q}{|a|} \ll Q^{1+\alpha} \log Q \ll \epsilon Q^{7/4+\epsilon}. \]

(4.4)

On the other hand (4.3) leads to
\[ \sum_{Q^n < |c| \leq Q/y_0} \mathcal{E}_{c,N} \ll \epsilon z_0 Q^{7/4} \log Q + Q^{5/4} \sum_{1 \leq |c| \leq Q} c^{-1/2 + \epsilon} + Q^2 \sum_{c > Q^n} c^{-3/2 + \epsilon} \ll \epsilon Q^{7/4+\epsilon} + Q^{5/4 + 1/2 + \epsilon} + Q^{2 + \alpha(-1/2 + \epsilon)} \ll Q^{7/4 + \epsilon}. \]

(4.5)

From (4.1)–(4.5) we now infer
\[ \mathfrak{N}_Q(N, z_0; \beta) = \frac{1}{N^2} \sum_{Q^n \leq |c| \leq Q/y_0} \Phi_N(c) F(c) + O_{\epsilon,N,z_0} \left( Q^{7/4 + \epsilon} \right). \]

(4.6)

Using \( I_c \subseteq \left[ - \sqrt{Q^2 - c^2 y_0^2}, \sqrt{Q^2 - c^2 y_0^2} \right] \) and the change of variable \( u = C \tan x \) we get
\[ F(c) = 2y_0 \int_{I_c} \sqrt{\frac{Q^2}{c^2 y_0^2 + (a - cx_0)^2} - 1} \, da \leq 4y_0 \int_0^\sqrt{Q^2 - C^2} \sqrt{\frac{Q^2}{C^2 + u^2} - 1} \, du \]
\[ = 4y_0 \int_0^{\arctan \sqrt{Q^2/C^2 - 1}} \sqrt{Q^2 - C^2 \cos^2 x} \, dx \leq 4y_0 Q \int_0^{\arctan \sqrt{Q^2/C^2 - 1}} \frac{dx}{\cos x} \]
\[ = 2y_0 Q \log \frac{1 + \sin x}{1 - \sin x} \bigg| _{x = 0}^{\arctan \sqrt{Q^2/C^2 - 1}} = 4y_0 Q \log \left( \frac{Q}{C} + \sqrt{\frac{Q^2}{C^2 - 1}} \right) \ll z_0 Q \log Q. \]

The total variation of \( F \) on \( \left[ - \frac{Q}{y_0}, -Q^3 \right] \) and on \( \left[ Q^3, \frac{Q}{y_0} \right] \) is also \( \ll z_0 Q \log Q \) because \( F \) is slowly oscillating. Applying Corollary 4 to the sum from (4.6) we now infer
\[ \mathfrak{N}_Q(N, z_0; \beta) = \frac{C_N}{N^3} \int_{Q^n \leq |c| \leq Q/y_0} F(c) \, dc + O_{\epsilon,N,z_0} \left( Q^{7/4 + \epsilon} \right) \]
\[ = \frac{C_N}{N^3} \int_{-Q/y_0}^{Q/y_0} F(c) \, dc + O_{\epsilon,N,z_0} \left( Q^{7/4 + \epsilon} \right). \]
Using the substitution \( c = \frac{Qu}{y_0} \), \( a = Qv + c \xi_0 = (v + \frac{wx_0}{y_0})Q \), the integral in the main term above is evaluated as
\[
\int_{-Q/y_0}^{Q/y_0} F(c) \, dc = 2 \int_{-Q/y_0}^{Q/y_0} \int_{\mathcal{C}} f(c, a) \, da \, dc
\]
\[
= 2 \int_{u^2 + v^2 \leq 1} \frac{1}{u^2 + v^2 - 1} \, du \, dv + 2 \int_{u^2 + v^2 < 1} \frac{1}{u^2 + v^2 - 1} \, du \, dv
\]
\[
= 2 \int_{0}^{1} \int_{-\pi/2}^{\pi/2} \sqrt{1 - r^2} \, d\theta \, dr + 2 \int_{0}^{1} \int_{\pi/2}^{\pi + \arctan \beta} \sqrt{1 - r^2} \, d\theta \, dr
\]
\[
= \pi (\pi + 2 \arctan \beta).
\]
This completes the proof of the proposition. \( \square \)

Taking stock on (2.9) we obtain (recall that \( Q^2 = e^R + O(e^{-R}) \))
\[
\# \Gamma(N)_{z_0, R} = \# \{ \gamma \in \Gamma(N) : A^2 + B^2 + C^2 + D^2 \leq Q^2 \} = \Re \sqrt{Q^2 + O_{z_0}(Q)}(N, z_0; \infty) \]
\[
= \frac{\pi^2 C N Q^2}{N^3} + O_{e, N, z_0}(Q^{7/4 + \varepsilon}) = \frac{6Q^2}{\Gamma(1) : \Gamma(N)} + O_{e, N, z_0}(Q^{7/4 + \varepsilon})
\]
\[
= \frac{6e^R}{\Gamma(1) : \Gamma(N)} + O_{e, N, z_0} \left( e^{(7/8 + \varepsilon) R} \right),
\]
which proves Corollary 2.

**Proof of Theorem 1.** Set \( \mathfrak{N}_Q(\beta) = \mathfrak{N}_Q(N, z_0; \beta) \). As a consequence of Proposition 2 and of the inequality \( |\arctan(\beta + \beta_0) - \arctan \beta| \leq |\beta_0| \) we have
\[
|\mathfrak{N}_Q(\beta + \beta_0) - \mathfrak{N}_Q(\beta)| \ll_{e, N, z_0} Q^2 |\beta_0| + Q^{7/4 + \varepsilon}.
\]

Let \( S_Q(\omega) = S_Q(N, z_0, z_1; \omega) \) denote the cardinality of the set of \( \gamma \in \Gamma(N) \) with \( A^2 + B^2 + C^2 + D^2 \leq Q^2 \) and \( -\frac{\pi}{2} \leq \theta_{z_0, z_1}(\gamma) \leq \omega \). Partitioning this set according to whether or not \( \min\{|A|, |C|\} > Q^\alpha \) and employing Lemma 1 we find that, up to an error \( \ll_{z_0} Q^{1 + \alpha} \log Q \), \( S_Q(\omega) \) equals
\[
\# \{ \gamma \in \Gamma(N) : A^2 + B^2 + C^2 + D^2 \leq Q^2, |A|, |C| > Q^\alpha, -\pi/2 \leq \theta_{z_0, z_1}(\gamma) \leq \omega \}.
\]
By (2.9) there is \( K_2 = K_2(z_0) > 0 \) such that the number in (4.9) is
\[
\leq \# \{ \gamma \in \Gamma(N) : (C^2 + A^2) \left( 1 + \frac{D^2}{C^2} \right) \leq Q_1^2, |A|, |C| > Q^\alpha, -\frac{\pi}{2} \leq \theta_{z_0, z_1}(\gamma) \leq \omega \},
\]
where we set \( Q_1 := \sqrt{Q^2 + K_2 Q} = Q + O_{z_0}(1) \). According to (2.8) the number in (4.10) is
\[
\leq \# \{ \gamma \in \Gamma(N) : (C^2 + A^2) \left( 1 + \frac{D^2}{C^2} \right) \leq Q_1^2, \frac{A}{C} \in x_+ + \mathfrak{S}(y_\ast, \tan \omega) + \left[ -\frac{3K_1}{Q_{2\alpha}^2}, \frac{3K_1}{Q_{2\alpha}} \right] \}.
\]
Taking \( \alpha = \frac{1}{2} \) and applying (4.8) to \( |\beta_0| = Q^{-2\alpha} = Q^{-1/4} \) we find
\[
S_Q(\omega) \leq \# \{ \gamma \in \Gamma(N) : (C^2 + A^2) \left( 1 + \frac{D^2}{C^2} \right) \leq Q_1^2, \frac{A}{C} \in x_+ + \mathfrak{S}(y_\ast, \tan \omega) \}
\]
\[
+ O_{e, N, z_0, z_1}(Q^{7/4 + \varepsilon}).
\]
The number of matrices \( \gamma \in \Gamma(N) \) for which \( \frac{A}{C} = \mu \) and \( A^2 + B^2 + C^2 + D^2 \leq Q^2 \) is \( \ll_{z_0, \mu} Q \) as \( Q \to \infty \). Using this fact together with (2.7), (2.9), and (2.10), we find that, up to a term of
order \( O_{z_0}(Q_1) = O_{z_0}(Q) \), the main term in the right-hand side of (4.11) is given by

\[
\begin{cases}
\mathcal{N}_{Q_1}(x_0 - y_0 \cot \omega) + \mathcal{N}_{Q_1}(x_0 + y_0 \tan \omega) - \mathcal{N}_{Q_1}(x_0 - y_0) \\
+ \mathcal{N}_{Q_1}(\infty) - \mathcal{N}_{Q_1}(x_0 + y_0) & \text{if } \omega > 0,
\end{cases}
\]

\[
\begin{cases}
\mathcal{N}_{Q_1}(x_0) - \mathcal{N}_{Q_1}(x_0 - y_0) + \mathcal{N}_{Q_1}(\infty) - \mathcal{N}_{Q_1}(x_0 + y_0) & \text{if } \omega = 0,
\end{cases}
\]

\[
\begin{cases}
\mathcal{N}_{Q_1}(x_0 + y_0 \tan \omega) - \mathcal{N}_{Q_1}(x_0 - y_0) & \text{if } \omega < 0,
\end{cases}
\]

(4.12)

\[
\begin{align*}
= & \mathcal{N}_{Q_1}(x_0 + y_0 \tan \omega) + \mathcal{N}_{Q_1}(x_0 - y_0 \cot \omega) - \mathcal{N}_{Q_1}(x_0 + y_0)
- & \mathcal{N}_{Q_1}(x_0 - y_0) + \begin{cases}
\pi & \text{if } \omega > 0, \\
0 & \text{if } \omega < 0.
\end{cases}
\end{align*}
\]

As a result of Proposition 2 and \( Q_1 = Q + O_{z_0}(1) \) the expression in (4.12) equals

\[
\frac{\pi C N^2}{N^3} \left( \arctan(x_0 + y_0 \tan \omega) + \arctan(x_0 - y_0 \cot \omega) - \arctan(x_0 + y_0)
- \arctan(x_0 - y_0) + \begin{cases}
\pi & \text{if } \omega > 0, \\
0 & \text{if } \omega < 0,
\end{cases}\right) + O_{\epsilon,N,z_0,z_1}(Q^{7/4+\varepsilon}).
\]

Letting \( \Xi_{x_0,y_0} \) as in (1.4) we now infer

\[
S_Q(\omega) \leq \frac{\pi^2 C N^2 \Xi_{x_0,y_0}(\omega)}{N^3} Q^2 + O_{\epsilon,N,z_0,z_1}(Q^{7/4+\varepsilon}).
\]

The opposite inequality

\[
S_Q(\omega) \geq \frac{\pi^2 C N^2 \Xi_{x_0,y_0}(\omega)}{N^3} Q^2 + O_{\epsilon,N,z_0,z_1}(Q^{7/4+\varepsilon})
\]

is derived in a similar way. Therefore equality holds in (4.13). Equality (1.5) now follows taking \( Q^2 = 2 \cosh R = e^R + e^{-R} \).

Estimates (1.5) and (4.7) provide

\[
P_{\Gamma(N),B(z_0,R),z_0,z_1}(\omega) = \frac{\# \{ \gamma \in \Gamma(N)_{z_0,R} : -\pi/2 \leq \theta_{z_0,z_1}(\gamma) \leq \omega \}}{\# \Gamma(N)_{z_0,R}}
\]

(4.14)

\[
= \frac{\frac{\pi^2 C N^2 \Xi_{x_0,y_0}(\omega)}{N^3} e^R + O_{\epsilon,N,z_0,z_1}(e^{(7/8+\varepsilon)R})}{\frac{\pi^2 C N^2 e^R + O_{\epsilon,N,z_0,z_1}(e^{(7/8+\varepsilon)R})}{\Xi_{x_0,y_0}(\omega) + O_{\epsilon,N,z_0,z_1}(e^{-1/8+\varepsilon)R})}.
\]

The function \( \Xi_{x_0,y_0}(\omega) \) is differentiable on \([-\pi/2, \pi/2]\) with

\[
\Xi'_{x_0,y_0}(\omega) = \frac{y_0}{2\pi \cos^2 \frac{\omega}{2}(1 + (x_0 + y_0 \tan \frac{\omega}{2})^2)} + \frac{y_0}{2\pi \sin^2 \frac{\omega}{2}(1 + (x_0 - y_0 \cot \frac{\omega}{2})^2)}
\]

(4.15)

\[
= y_0 \left( \frac{1}{\cos^2 \frac{\omega}{2} + (x_0 \cos \frac{\omega}{2} + y_0 \sin \frac{\omega}{2})^2} + \frac{1}{\sin^2 \frac{\omega}{2} + (x_0 \sin \frac{\omega}{2} - y_0 \cos \frac{\omega}{2})^2} \right)
\]

\[
= \frac{2}{\pi} \cdot \frac{y_0 (1 + y_0^2)}{(1 + x_0^2 + y_0^2) - ((1 + x_0^2 - y_0^2) \cos \omega + 2x_0y_0 \sin \omega)^2}
\]

\[
= \frac{1}{\pi^2} \theta_{z_0,z_1}(\omega).
\]

The second part of Theorem 1 now follows from (4.14) and (4.15).
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