§16.5 CURL AND DIVERGENCE

Recall that given a vector field \( \mathbf{F} = \mathbf{P}(x,y,z), \mathbf{F} = \mathbf{P} \mathbf{i} + \mathbf{Q} \mathbf{j} + \mathbf{R} \mathbf{k} \), we defined

\[
\text{curl } \mathbf{F} = \begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial Q}{\partial z} & \frac{\partial R}{\partial x} & \frac{\partial P}{\partial y} \\
\frac{\partial R}{\partial y} & \frac{\partial P}{\partial z} & \frac{\partial Q}{\partial x}
\end{vmatrix}
\text{ and div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.
\]

and noticed that:

1) \( \mathbf{F} \) conservative \( \implies \) curl \( \mathbf{F} = \mathbf{0} \)

2) curl \( \mathbf{F} = \mathbf{0} \) on \( \mathbb{R}^3 \) \( \implies \) \( \mathbf{F} \) conservative

A vector field with curl \( \mathbf{F} = \mathbf{0} \) is called \textit{irrotational}.

Furthermore, we always have

3) \( \text{div} \left( \text{curl } \mathbf{F} \right) = 0 \) if \( \mathbf{F} \) is a \( C^2 \) vector field.

This is because

\[
\text{div} \left( \text{curl } \mathbf{F} \right) = \text{div} \left( \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial Q}{\partial z} & \frac{\partial R}{\partial x} & \frac{\partial P}{\partial y} \\
\frac{\partial R}{\partial y} & \frac{\partial P}{\partial z} & \frac{\partial Q}{\partial x}
\end{vmatrix} \right) = \text{div} \left( \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \right) = 0
\]

A vector field with \( \text{div } \mathbf{F} = 0 \) is called \textit{incompressible}.

For every \( C^2 \) function \( f = f(x,y,z) \), we have

\[
\text{div} \left( \nabla f \right) = \nabla \cdot \nabla f = \left\langle \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2} \right\rangle \cdot \left\langle f_x, f_y, f_z \right\rangle = f_{xx} + f_{yy} + f_{zz} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f.
\]

The differential operator \( \Delta \) (sometimes denoted \( \nabla^2 \)) is called the \textit{Laplace operator} (or simply \textit{laplacian}) on \( \mathbb{R}^3 \).

Two examples:

1) \textbf{Gravitational Field} \( \mathbf{F}(x,y,z) = -Mm G \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle \),

where \( r = r(x,y,z) = \sqrt{x^2 + y^2 + z^2} \), so \( \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r} \).
\( \mathbb{F} : \mathbb{R}^3 \setminus \{(0,0,0)\} = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z) \neq (0,0,0)\} \to \mathbb{R}^3 \) was seen to be conservative on \( \mathbb{R}^3 \setminus \{(0,0,0)\} \), thus \( \text{curl} \ \mathbb{F} = 0 \).

The latter can also be checked directly:

\[
\text{curl} \ \mathbb{F} = -M \cdot m \cdot g \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x^2 & y^2 & z^2 \\
x r^3 & y r^3 & -z r^3
\end{vmatrix}
\]

\[
= -M m g \left( \frac{\partial}{\partial y} (z r^{-3}) - \frac{\partial}{\partial z} (y r^{-3}) \right)
- \frac{\partial}{\partial x} (x r^{-3}) - \frac{\partial}{\partial y} (y r^{-3}) - \frac{\partial}{\partial z} (z r^{-3})
\]

\[
= -3 r^{-4} \left( 2 \frac{x}{\partial y} - y \frac{\partial}{\partial z} \right) = 0.
\]

One checks similarly that the 2nd and 3rd components of \( \text{curl} \ \mathbb{F} \) are 0.

2) Determine whether \( \mathbb{F}(x,y,z) = (e^{x \sin(yz)}, z e^{x \cos(yz)}, y e^{x \cos(yz)}) \) is conservative. If yes, find a function \( f = f(x,y,z) \) s.t. \( \mathbb{F} = \nabla f \).

\[
\text{curl} \ \mathbb{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
e^{x \sin(yz)} & z e^{x \cos(yz)} & y e^{x \cos(yz)}
\end{vmatrix}
\]

\[
= \left( e^{x \cos(yz)} - y z e^{x \sin(yz)} - e^{x \cos(yz)} + 2 ye^{x \cos(yz)},
-ye^{x \cos(yz)} - ye^{x \cos(yz)},
z e^{x \cos(yz)} - z e^{x \cos(yz)} \right)
\]

\[
= (0,0,0) = 0 \quad \text{on} \ \mathbb{R}^3 \quad \Rightarrow \ \mathbb{F} \ \text{conservative on} \ \mathbb{R}^3
\]

The function \( f \) must satisfy:

\[
\begin{align*}
\frac{df}{dx} &= e^{x \sin(yz)} \quad \Rightarrow \quad f(x,y,z) = \int e^{x \sin(yz)} \, dx + g(y,z) = e^{x \sin(yz)} + g(y,z) \\
\frac{df}{dy} &= z e^{x \cos(yz)} \quad \Rightarrow \quad \frac{df}{dy} = z e^{x \cos(yz)} + \frac{\partial}{\partial y} \Rightarrow \frac{\partial}{\partial y} = 0 \\
\frac{df}{dz} &= y e^{x \cos(yz)} \quad \Rightarrow \quad g(y,z) = h(z) \Rightarrow f(x,y,z) = e^{x \sin(yz)} + h(z)
\end{align*}
\]

\[
\frac{df}{dz} = ye^{x \cos(yz)} + h'(z) 
\]

\[
h'(z) = 0 \quad \Rightarrow \quad f(x,y,z) = e^{x \sin(yz)} + C.
\]
§16.9 STOKES THEOREM

Suppose: \( S \) oriented smooth surface bounded by a simple, closed, piecewise \( C^1 \) curve \( \partial S = \partial S \), called the boundary of \( S \).

The (positive) orientation on \( S \) induces a (positive) orientation on \( \partial S \): while walking along \( \partial S \) with your head in the normal direction, will always have the surface \( S \) on your left.

\[
\int_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \int_S (\text{curl} \mathbf{F}) \cdot d\mathbf{S}
\]

\(\leftarrow\) STOKES FORMULA for surfaces with boundary

Remarks

1) \[ \int_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} P \, dx + Q \, dy + R \, dz \quad \text{if} \quad \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}. \]

2) GREEN is a particular case of STOKES.

To see this, regard \( D \subset \mathbb{R}^2 \) as a surface \( S = \{(x,y,0) | (x,y) \in D\} \subset \mathbb{R}^3 \).

Let \( \mathbf{F}(x,y) = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j} \) be a vector field on \( D \), viewed as a vector field \( \mathbf{F}(x,y,2) = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j} + 0 \mathbf{k} \) on the "full cylinder" \( \{(x,y, z) | (x,y) \in D\} \) over \( D \), which contains \( S \). The normal to \( S \) is \( \mathbf{n} = \mathbf{k} \).

\( R_x \times \mathbf{n} = \mathbf{l} \times \mathbf{k} = \mathbf{j}, \quad \text{curl} \mathbf{F} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial P}{\partial y} & \frac{\partial Q}{\partial x} & \frac{\partial R}{\partial y} \\ \frac{\partial P}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial R}{\partial x} \end{array} \right| = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle \)

\[ \Rightarrow \int_S (\text{curl} \mathbf{F}) \cdot d\mathbf{S} = \int_D \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle \cdot \langle 0, 0, 1 \rangle \, dA = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \\
\int_{\partial S} P \, dx + Q \, dy + R \, dz = \int_{\partial D} P \, dx + Q \, dy. \]