§16.1 VECTOR FIELDS

DEFINITION A vector field on \( \mathbb{R}^n \) is a function \( \mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n \).

- For \( n=2 \) \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} = \langle P, Q \rangle \) with \( P = P(x, y) \), \( Q = Q(x, y) \) scalar valued functions.
- For \( n=3 \) \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} = \langle P, Q, R \rangle \) with \( P = P(x, y, z) \) etc.

A field \( \mathbf{F} \) is conservative when it is a gradient field, that is there exists \( \mathbf{F} = \nabla f \) on \( D \)

\[
\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad \text{and} \quad \frac{\partial f}{\partial z} = R.
\]

In this case \( f \) is called the potential function for \( \mathbf{F} \).

EXAMPLES
1) \( \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2 \), \( \mathbf{F}(x, y) = \langle -y, x \rangle \).

\[ |\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = |\langle x, y \rangle| \quad \text{and} \quad \mathbf{F}(x, y) \cdot \langle -y, x \rangle = -xy + xy = 0 \]

Is the field \( \mathbf{F} \) conservative?

Suppose YES. Then there exists \( f : \mathbb{R}^2 \to \mathbb{R} \) s.t.

\[ \mathbf{F} = \nabla f, \quad \text{that is} \]

\[
\begin{aligned}
\frac{\partial f}{\partial x} &= -y \\
\frac{\partial f}{\partial y} &= x \\
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= -1 \\
\frac{\partial^2 f}{\partial y \partial x} &= 1 \\
\end{aligned}
\]

Contradiction! (CLAIMED)

Therefore \( \mathbf{F} \) is NOT conservative.

2) **Gravitational Field**

\( (0,0,0) \) \( \to \) Mass \( M \)

\( \to \) Mass \( m \)

\( \mathbf{F}(x, y, z) \) gravitational force

**Newton's Gravitational Law:** \( |\mathbf{F}| = \frac{G M m}{r^2} \), where \( r = |\mathbf{r}| \)

(The attraction force is inversely proportional to the square of the distance \( r = r(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) between the two bodies).
\[- \vec{r} \over \vec{r} \] is the unit direction for \( \vec{F} \) (as \( \vec{F} \) points toward \((0,0,0)\))

\[
\vec{F} = -|\vec{F}| \frac{\vec{r}}{r^3} = -\frac{MmG}{r^3} \vec{r} = -MmG \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)
\]

\[
\begin{align*}
\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \quad \Rightarrow \quad \frac{\partial f}{\partial x} (\frac{1}{r}) = -\frac{2x}{r^3} = -\frac{x}{r^3} \\
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r} \quad \Rightarrow \quad \frac{\partial f}{\partial y} (\frac{1}{r}) = -\frac{y}{r^3} \\
\frac{\partial r}{\partial z} &= \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r} \quad \Rightarrow \quad \frac{\partial f}{\partial z} (\frac{1}{r}) = -\frac{z}{r^3}
\end{align*}
\]

So \( \nabla f = -\vec{F} \) where \( f(x,y,z) = \frac{MmG}{r} = \frac{MmG}{\sqrt{x^2+y^2+z^2}} \). This shows that the gravitational field is conservative.

§ 16.2 Line integrals

Aim: generalize the Riemann integral \( \int_a^b f(t) \, dt \) replacing the segment \([a,b]\) by a smooth curve \( C \) in \( \mathbb{R}^3 \) (or in \( \mathbb{R}^2 \)) parameterized by \( \vec{r}(t) = (x(t), y(t), z(t)) \), \( t \in [a,b] \) and taking \( f \) continuous on \( C = \vec{r}([a,b]) \). When \( C \subset \mathbb{R}^2 \) just drop the third coordinate.

Recall that \( ds = |r'(t)| \, dt \). Define

\[
\int_C f(x,y,z) \, ds = \int_a^b f(\vec{r}(t)) |r'(t)| \, dt
\]

\[
= \int_a^b f(x(t), y(t), z(t)) \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} \, dt
\]

\[
\int_C f(x,y,z) \, dx = \int_a^b f(\vec{r}(t)) x'(t) \, dt \quad \text{and similarly for}
\]

\[
\int_C f(x,y,z) \, dy \quad \text{and} \quad \int_C f(x,y,z) \, dz.
\]

Example 1) Evaluate \( \int_C x \, dy \) where \( C \) is the line segment from \((0,0)\) to \((4,6)\).
$C$ is parameterized by $\vec{r}(t) = \langle 0, 3 \rangle + t \langle 4, 3 \rangle = \langle 4t, 3+3t \rangle$, $t \in [0, 1]$.

\[
\begin{align*}
\begin{cases}
X(t) = 4t \\
y(t) = 3 + 3t
\end{cases}
\quad ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt = \sqrt{4^2 + 3^2} \, dt = 5 \, dt
\end{align*}
\]

\[
\int_C x \sin y \, ds = \int_0^1 4t \sin(3+3t) \cdot 5 \, dt = 20 \int_0^1 t \sin(3+3t) \, dt
\]

\[
= 20 \left[ \int_0^6 \frac{u-3}{3} \sin u \, du \right] = \frac{20}{9} \left[ \int_0^6 (u-3) \sin u \, du \right]
\]

(can integrate by parts)

2) Evaluate $\int_C x \sin(y+2) \, ds$ where $C$ has parametric equations $x = t^2$, $y = t^3$, $z = t^4$, $t \in [0, 5]$.

\[
\begin{align*}
\int_C x \sin(y+2) \, ds &= \int_0^5 t^2 \sin(t^3 + t^4) \sqrt{4 + 9t^2 + 16t^4} \, dt
\end{align*}
\]

3) Evaluate $I = \int_C y \, dx + z \, dy + x \, dz$ where $C: x = \sqrt{t}, y = t, z = t^2$, $t \in [1, 4]$.

\[
I = \int_1^4 \left( t \frac{d(\sqrt{t})}{dt} + t^2 \frac{dt}{dt} + \sqrt{t} \frac{d(t^2)}{dt} \right) \, dt
\]

\[
= \int_1^4 \left( \frac{\sqrt{t}}{2} + t^2 + 2t \sqrt{t} \right) \, dt
\]

INTEGRATING VECTOR FIELDS ON CURVES

DEFINITION The line integral of a continuous vector field $\vec{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ ($n = 2$ or $n = 3$) on a smooth curve $C$ parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a, b]$ is defined by

\[
\int_C \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

independent of parameterization for simple parameterizations of $C$.
If $\mathbf{F} = \mathbf{F}(x,y,z)$ represents a force field, then the integral

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

is the work done by $\mathbf{F}$ along the curve $\mathcal{C}$.

**Remark**

![Diagram with $\mathbf{F}(x,y,z)$ tangent unit vector at $P(x,y,z)$]

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{1} \left( \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)| \, dt = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \, ds$$

**Example** Find the work done by the force field $\mathbf{F} = (x-y^2, y^2-z^2, z-x)$ on a particle that moves along the line segment $\mathcal{C}$ from $(0,0,1)$ to $(2,1,0)$.

$\mathcal{C}$ is parameterized by $\mathbf{r}(t) = (0,0,1) + t(2,1,0) = (2t, t, 1-t)$, $t \in [0,1]$.

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_{0}^{1} (2t-t^2, t-(1-t)^2, 1-t-(2t)^2) \cdot (2,1,0) \, dt = ... = \frac{7}{3}.$$