LECTURE 13 (09/25/15)

§14.7 MAXIMUM AND MINIMUM VALUES

**Definition** A function of \( n \) variables \( D \xrightarrow{f} \mathbb{R} \) has a

local minimum (respectively local maximum) at \((a_1, \ldots, a_n) \in D\) if

\[
\begin{align*}
  f(x_1, \ldots, x_n) &\geq f(a_1, \ldots, a_n) = m \quad \text{for every } (x_1, \ldots, x_n) \in D \text{ near } (a_1, \ldots, a_n) \\
  \text{(resp. } f(x_1, \ldots, x_n) &\leq f(a_1, \ldots, a_n) = M \text{ for every } (x_1, \ldots, x_n) \in D \text{ near } (a_1, \ldots, a_n))
\end{align*}
\]

\( m \) is called the local minimum value of \( f \).
\( M \) is called the local maximum value of \( f \).

When these inequalities hold for every \((x_1, \ldots, x_n) \in D\), we say that \( f \) has an absolute (or global) minimum (respectively an absolute maximum) at \((a_1, \ldots, a_n)\).

Points of local min or local max are called local extrema.

**Definition** \( a = (a_1, \ldots, a_n) \) is a critical point for \( f = f(x_1, \ldots, x_n) \) if \( \nabla f(a) = 0 \) (\( \iff f_x(a) = \cdots = f_{x_n}(a) = 0 \)) or if at least one of \( f_x(a), \ldots, f_{x_n}(a) \) DNE. A point \( a \) with \( \nabla f(a) = 0 \) is called a stationary point for \( f \).

Recap: **Fermat's Theorem** (from Calculus I) says that

\[
\begin{align*}
  &\text{if } \int_{\mathbb{R}^n} f(x) \, dx \\
  &\text{a is a local extremum point for } f = f(x) \\
  \text{then } f(x) = x^3, a = 0, f'(0) = 0 \quad \text{critical but not local extremum point}
\end{align*}
\]

In \( n \geq 2 \) variables, if \((a_1, \ldots, a_n)\) is a local extremum for \( f = f(x_1, \ldots, x_n)\), then \( a_2 \) is a local extremum for \( x_2 \rightarrow f(x_2, a_2, \ldots, a_n) \), so by Fermat's Theorem \( f_{x_2}(a_1, a_2, \ldots, a_n) = 0 \) or DNE, etc.
Hence we have:

**THEOREM** \((a_1, \ldots, a_n)\) local extremum point for \(f \implies (a_1, \ldots, a_n)\), critical point for \(f\).

**EXAMPLE** \(f(x, y) = x^2 - y^2\)

\((\nabla f)(x, y) = \langle 2x, -2y \rangle\)

Critical points:
\[
\begin{cases}
2x = 0 \\
-2y = 0
\end{cases}
\]

\((0, 0)\) the only critical point.

For \(\varepsilon > 0\) arbitrarily close to 0:

\(f(\varepsilon, 0) = \varepsilon^2 > f(0, 0) = 0 \implies (0, 0)\) NOT local MAX

\(f(0, \varepsilon) = -\varepsilon^2 < f(0, 0) = 0 \implies (0, 0)\) NOT local MIN

\((0, 0)\) not local extremum.

Such a point (\(\nabla f)(x, y) = 0\) but \((x, y)\) is not local extremum point

is called *saddle point*.

**THE SECOND DERIVATIVE TEST**

Suppose that:

(i) \(f = f(x, y)\) has continuous 2nd order derivatives near \((a, b)\).

(ii) \((\nabla f)(a, b) = (0, 0)\).

Consider \(D = D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2\).

Then:

1. \(D > 0 \land f_{xx}(a, b) > 0 \implies f\) has local MIN at \((a, b)\)
2. \(D > 0 \land f_{xx}(a, b) < 0 \implies f\) has local MAX at \((a, b)\)
3. \(D < 0 \implies (a, b)\) saddle point.
4. In the other situations (including \(D = 0\)) the test is inconclusive.

**REMARK** \(D > 0 \land f_{yy}(a, b) > 0 \implies f\) has local MIN at \((a, b)\)

\(D > 0 \land f_{yy}(a, b) < 0 \implies f\) has local MAX at \((a, b)\).
EXAMPLES

1) \( f(x, y) = e^y(y^2 - x^2) \)

\[
\begin{align*}
  f_x &= -2x e^y, \quad f_y = e^y(y^2 - x^2) + e^y \cdot 2y = e^y(y^2 + 2y - x^2) \\
  f_{xx} &= -2e^y, \quad f_{xy} = -2xe^y, \quad f_{yy} = e^y(y^2 + 2y - x^2) + e^y(2y+2) \\
  &= e^y(y^2 + 4y - x^2 + 2)
\end{align*}
\]

Critical points:

\[
\begin{align*}
  -2xe^y &= 0 \quad \iff \quad \{x=0\} \\
  e^y(y^2 + 2y - x^2) &= 0 \quad \iff \quad \{y^2 + 2y - x^2 = 0\}
\end{align*}
\]

\[
\begin{align*}
  \{x=0\} &\quad \iff \quad \{(x, y) \in \{(0, -2), (0, 0)\}\} \\
  y(y+2) &= 0
\end{align*}
\]

\[
D(0, -2) = \begin{vmatrix} -2e^2 & 0 \\ 0 & -2e^2 \end{vmatrix} = 4e^{-4} > 0
\]

\[
f_{xx}(0, -2) < 0 \quad \Rightarrow \quad (0, -2) \text{ local max}
\]

\[
D(0, 0) = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0
\]

\[
\Rightarrow \quad (0, 0) \text{ saddle point.}
\]

2) \( f(x, y) = (1+xy)(x+y) = x + y + x^2y + xy^2 \)

\[
\begin{align*}
  f_x &= 1 + 2xy + y^2, \quad f_y = 1 + 2xy + x^2 \\
  f_{xx} &= 2y, \quad f_{xy} = 2x, \quad f_{yy} = 2x + 2y
\end{align*}
\]

Critical points:

\[
\begin{align*}
  1 + 2xy + y^2 &= 0 \quad \iff \quad y^2 - x^2 = 0 \\
  1 + 2xy + x^2 &= 0 \quad \iff \quad \{y = x, y = -x\}
\end{align*}
\]

\[
y = x \Rightarrow 1 + 3x^2 = 0 \quad \Rightarrow \quad \text{no solution}
\]

\[
y = -x \Rightarrow 1 - 2x^2 + x^2 = 0 \quad \iff \quad x^2 = 1 \quad \iff \quad x = \pm 1
\]

\[
\Rightarrow \quad (x, y) \in \{(-1, 1), (1, -1)\}
\]

\[
D(-1, 1) = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0 \quad \Rightarrow \quad (-1, 1) \text{ saddle point}
\]

\[
D(1, -1) = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = 4 < 0 \quad \Rightarrow \quad (1, -1) \text{ saddle point}
\]