§14.4 PARTIAL DERIVATIVES (continued)

**Definition** The **gradient** of a function \( D \xrightarrow{f} \mathbb{R} \) is defined as the vector-valued function (called vector field)

\[
\nabla f : = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right),
\]

or \((\nabla f)(a_1, \ldots, a_n) = \left( \frac{\partial f}{\partial x_1}(a_1, \ldots, a_n), \ldots, \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n) \right)\).

The equation of the tangent plane to the surface \( z = f(x,y) \) at the point \( P_0 (a, b, c = f(a,b)) \) becomes

\[
\boxed{z - c = \langle \nabla f \rangle(a, b) \cdot \langle x - a, y - b \rangle}
\]

\[
\iff z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)
\]

\(L(x,y)\) the linearization of \( f(x,y) \) at \((a, b)\).

The **increment of** \( z \) is \( \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) \).

**Definition** The function \( f = f(x,y) \) is **differentiable** at \((a, b)\) if

\[
\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,
\]

with \((\varepsilon_1, \varepsilon_2) \to (0,0) \iff (\Delta x, \Delta y) \to (0,0)\).

This means that

\[
f(a + \Delta x, b + \Delta y) = L(x,y) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,
\]

that is the linear approximation is a good approximation when \((x,y)\) approaches \((a,b)\).

**Theorem** \( f \) is differentiable at \((a, b)\) whenever \( f_x \) and \( f_y \) exist near \((a, b)\) and are continuous at \((a, b)\).

**Example** \( f(x,y) = \frac{x}{x+y} \), \((a, b) = (2,1)\)

\[
f_x = \frac{2}{x(x+y)} = \frac{x+y-x}{(x+y)^2} = \frac{y}{(x+y)^2} \text{ and}
\]
\[ f_y = \frac{\partial}{\partial y} \left( \frac{x}{x+y} \right) = -\frac{x}{(x+y)^2} \quad \text{exist near (2,1) and are} \]
continuous at (2,1) \iff \text{differentiable at (2,1).}

Linearization
\[ L(x, y) = f(2,1) + \left( \frac{\partial f}{\partial x}(2,1), \frac{\partial f}{\partial y}(2,1) \right) \cdot (x-2, y-1) \]
\[ = \frac{1}{3} + \frac{x-2}{9} - \frac{2(y-1)}{9}. \]

Differentials
\[ n=1 \quad y = f(x) \quad \text{The differential of } y \text{ is defined as} \]
\[ dy = f'(x) \, dx \quad \text{(in this case } \frac{dy}{dx} = f'(x) = \frac{2y}{6x}) \]
with dx regarded as an infinitesimal variable.

\[ n=2 \quad z = f(x, y) \quad \text{The differential of } z \text{ is defined as} \]
\[ dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy = f_x \, dx + f_y \, dy, \]
with dx and dy independent infinitesimal variables.

\[ n=3 \quad w = f(x, y, z) \quad \text{The differential of } w \text{ is defined as} \]
\[ dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz = f_x \, dx + f_y \, dy + f_z \, dz, \]
with dx, dy and dz independent infinitesimal variables.

Geometric interpretation when \( z = f(x, y) \)

\[ \Delta z \]
\[ \Delta x = dx \]
\[ \Delta y = dy \]
\[ P(a, b, c) \quad c = f(a, b) \]
\[ R(a+\Delta x, b+\Delta y, c) \quad f(a, b) \]
\[ Q(a+\Delta x, b+\Delta y, c+\Delta z) \quad S \text{ is the surface defined by } z = f(x, y) \]
\[ \Delta z = \frac{\partial f}{\partial y}(a, b) \, dy \]
\[ \Delta x = x-a = dx; \quad \Delta y = y-b = dy \]
\[ T(a+\Delta x, b+\Delta y, c+\Delta z) \quad \text{TT is the tangent plane to } S \]
\[ \text{at } P: \quad z - c = f_x(a, b)(x-a) + f_y(a, b)(y-b) \]
\[ T \quad (a+\Delta x, b+\Delta y, c+\Delta z) \text{ belongs to } TT \text{ because} \]
\[ dz = \frac{\partial f}{\partial x}(a, b) \, dx + \frac{\partial f}{\partial y}(a, b) \, dy \]
\begin{align*}
\Delta z &= f(a+\Delta x, b+\Delta y) - f(a, b) \\
c &= f(a, b)
\end{align*}

\[ dz = \text{the change in height of the tangent plane} \]
\[ \Delta z = \text{the change in height of the surface} \]

\section*{§ 14.5 THE CHAIN RULE}

\textbf{In one variable:} \quad \{ \begin{align*}
y &= f(x) \\
x &= g(t)
\end{align*} \} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = f'(x(t))x'(t). \]

\textbf{In two variables:} \quad \{ \begin{align*}
z &= f(x, y) \\
x &= g(t), y &= h(t)
\end{align*} \} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{\partial z}{\partial t}

\text{with } f \text{ differentiable in } (x, y) \text{ and both } 
\text{and } \frac{dx}{dt} = \frac{\partial f}{\partial x} 
\text{and } \frac{dy}{dt} = \frac{\partial f}{\partial y} 
\text{differentiable in } t

\text{Case 2} \quad \{ \begin{align*}
z &= f(x, y) \\
x &= g(t), y &= h(t)
\end{align*} \} \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial z}{\partial t} \\
\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial y}
\end{pmatrix} = \begin{pmatrix} 0 \\
\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\end{pmatrix}

\text{with } f \text{ differentiable in } (x, y) \text{ and both } g \text{ and } h \text{ differentiable as functions in } (t, t)

\textbf{EXAMPLES} \quad 1) \quad z = \arctan \left( \frac{y}{x} \right) = f(x, y) \\
x = e^t, \quad y = 1 - e^{-t}

\begin{align*}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{y}{x} \right) = -\frac{1}{x^2 + y^2} \\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} \left( \frac{y}{x} \right) = \frac{x}{x^2 + y^2}
\end{align*} 

\begin{align*}
\Rightarrow \quad \frac{\partial z}{\partial t} &= \frac{-ye^{-t}}{x^2 + y^2} + \frac{xe^{-t}}{x^2 + y^2} = \frac{-e^{t}(1-e^{-t}) + e^{t}e^{-t}}{e^{2t} + (1-e^{-t})^2} = \frac{2-e^{t}}{e^{2t} + (1-e^{-t})^2}
\end{align*}