Lecture 7 (09/09/15)

§ 14.2 Limits and Continuity

Let \( x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \); then \( |x - a| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + ... + (x_n - a_n)^2} \)

\( x \to a \) in \( \mathbb{R}^n \) \( \iff \) \( |x - a| \to 0 \) in \( \mathbb{R} \)

Given a domain \( D \subseteq \mathbb{R}^n \), a point \( a = (a_1, ..., a_n) \in \mathbb{R}^n \) is called a cluster point of \( D \) in \( \mathbb{R}^n \) if \( D \) contains points that are arbitrarily close to \( a \).

Ex. 1) \((0,0)\) cluster point of \( D = \mathbb{R}^2 \setminus \{(0,0)\} \) \( \subseteq \) punctured plane

2) \((1,1)\) not a cluster point for \( D = \{(x,y) \mid x^2 + y^2 < 2\} \).

Def. Given:
- \( f : D \to \mathbb{R} \) function
- \( a \) a cluster point of \( D \)
- \( L \in \mathbb{R} \),

we write

\[
\lim_{x \to a} f(x) = L
\]

(in plain words "\( f(x) \) approaches \( L \) as \( x \) approaches \( a \)) if for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) s.t. if \( x \in D \) and \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \varepsilon \)

(\( \iff \) the distance between \( x \) and \( a \) is \( < \delta \) and \( x \neq a \))

Remark. This means that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) s.t. \( f \) maps the punctured ball of center \( a \) and radius \( \delta \) into the interval \((L-\varepsilon, L+\varepsilon)\). We think of \( \varepsilon \) as being arbitrarily small but \textbf{not} \( 0 \).

How to quantify \( \lim_{x \to a} f(x) = +\infty \) or \( \lim_{x \to a} f(x) = -\infty \). Replace \( L \) by \( \pm \infty \) and the "neighborhood" \((L-\varepsilon, L+\varepsilon)\) of \( L \)
by a neighborhood \((M, \infty)\) of \(+\infty\), respectively \((-\infty, M)\) of \(-\infty\).

**Def.** Given \(D, f, x, M\) as above

\[
\lim_{x \to a} f(x) = +\infty \iff \text{for each } M > 0, \text{ there exists } \delta = \delta(M) > 0 \text{ s.t. if } x \in D \text{ and } 0 < |x - a| < \delta, \text{ then } f(x) > M
\]

We think of \(M \to \infty\) being large.

\[
\lim_{x \to a} f(x) = -\infty \iff \text{for each } m < 0, \text{ there exists } \delta = \delta(m) > 0 \text{ s.t. if } x \in D \text{ and } 0 < |x - a| < \delta, \text{ then } f(x) < m.
\]

**Examples of limits using the \(\varepsilon - \delta\) (epsilon-delta) definition**

1. \(f(x) = |x|; \quad D = \mathbb{R}^n, \quad f : \mathbb{R}^n \to [0, \infty) \subset \mathbb{R}\).

Want to show \(\lim_{x \to a} |x| = |a|\).

\[\begin{align*}
\text{Will use the following inequality:} \\
\quad |f(x) - f(a)| \leq |x - a|,
\end{align*}\]

or equivalently

\[| |x| - |a| | \leq |x - a| . \]

Assuming \(\textcircled{2}\) holds true, let \(\varepsilon > 0\), then pick \(\delta = \delta(\varepsilon) = \varepsilon\).

Then \(|x - a| < \delta = \varepsilon \implies |f(x) - f(a)| < \varepsilon\), hence we established \(\textcircled{1}\). **Note:** we could also take \(\delta = \frac{\varepsilon}{2}, \delta = \varepsilon 10^{-12}\) etc.

**To see that \(\textcircled{2}\) holds, recap the triangle inequality**

\[|v + w| \leq |v| + |w| \text{ for every } v, w \in \mathbb{R}^n.\]

\[\begin{align*}
\text{If } & v = a, w = x - a \implies |x| \leq |a| + |x - a| \implies |x - a| \leq |x - a|; \\
\text{If } & v = x, w = a - x \implies |a| \leq |x| + |x - a| \implies |x - a| \leq |x - a|.
\end{align*}\]
\[ |x| - 1.5 \leq 15 - x. \]

2) \( \lim_{(x,y) \to (a,b)} \frac{x^2 - y^2}{x^2 + y^2} = ? \) \( D = \mathbb{R} \setminus \{(a,0)\} \)

**Case I.** \((a,b) \neq (0,0)\). In this case one can show that
\[
\lim_{(x,y) \to (a,b)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{a^2 - b^2}{a^2 + b^2} \quad \text{(expressing the "continuity" of \( f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \) at \((a,b)\)).}
\]

**Case II.** \((a,b) = (0,0)\).

Approaching \((0,0)\) along the line \(y = x\) we find
\[
L_1 = \lim_{x \to 0} f(x,x) = \lim_{x \to 0} \frac{x^2 - x^2}{x^2 + x^2} = \lim_{x \to 0} 0 = 0.
\]

Approaching \((0,0)\) along the line \(y = 2x\) we find
\[
L_2 = \lim_{x \to 0} f(x,2x) = \frac{x^2 - 4x^2}{x^2 + 4x^2} = \frac{1 - 4}{1 + 4} = -\frac{3}{5}
\]

\(L_1 \neq L_2 \implies \lim_{(x,y) \to (0,0)} f(x,y) \text{ DNE}\)

**Second approach**

Recap: polar coordinates
\[
\begin{align*}
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta
\end{cases}
\end{align*}
\]

\[
f(x,y) = \sqrt{x^2 + y^2} = \sqrt{r^2 (\cos^2 \theta + \sin^2 \theta)} = r \cos \theta \quad \text{when \( r \) takes different values while \( \theta \) takes different values}
\]

\[
\Rightarrow \lim_{(x,y) \to (0,0)} f(x,y) \text{ DNE}
\]

3) \( \lim_{(x,y) \to (0,0)} \frac{x^3 - y^3}{x^2 + y^2} ; \quad D = \mathbb{R} \setminus \{(0,0)\} \quad f(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \).
\[ f(x,y) = \frac{r^3 \cos^3 \theta - 3r \cos \theta \sin^2 \theta}{r^2} = r \left( \cos^3 \theta - 3\cos \theta \sin^2 \theta \right) \]

\[ -2 \leq \cos^3 \theta - \sin^3 \theta \leq 2 \implies 0 \leq |f(x,y)| \leq 2r \quad \text{(Squeeze Principle)} \]

\[ (x,y) \to (0,0) \]

\[ \lim_{(x,y) \to (0,0)} f(x,y) = 0. \]

\[ \varepsilon-\delta \text{ approach:} \quad \text{Let } \varepsilon > 0. \text{ The inequality } |f(x,y)| < \varepsilon \text{ is implied by } 2r < \varepsilon, \text{ so we can take } \delta = \frac{\varepsilon}{2} \text{ (or } \delta = \frac{\varepsilon}{10}, \delta = \varepsilon \cdot 10^{-12} \text{ etc.}). \text{ Then} \]

\[ r = |(x,y) - (0,0)| < \frac{\varepsilon}{2} \implies 2r < \varepsilon \implies |f(x,y)| = |f(x,y) - 0| < \varepsilon. \]