Lecture 2 (08/26/15)

§12.1 3D Coordinate Systems

§12.2 Vectors

§12.3 Dot Product (Inner Product)

**Aim:** explain the natural 1-1 correspondences

The set of points in the 3D space

\[ \mathbb{R}^3 = \{(a, b, c) | a, b, c \in \mathbb{R}\} \]

Point \( P \) in the physical 3D space \( \rightarrow \) triple \((a, b, c) \in \mathbb{R}^3\)

To realize such a correspondence we need:

1) An Origin (fixed point \( 0 \) in space)
2) Coordinates: three mutually perpendicular (orthogonal) lines through \( 0 \) with fixed directions.

Given a point \( P \), let:

- \( x = \) the directed distance from \( P \) to the \( yz \)-plane
- \( y = \) the directed distance from \( P \) to the \( xz \)-plane
- \( z = \) the directed distance from \( P \) to the \( xy \)-plane

In this figure \( a, b, c > 0 \)

\[ a, b, c \in \mathbb{R} \]

Express the length \( 10P1 \) as a function in \( a, b, c \)

**Pythagorean Theorem** \( \Rightarrow 10P1^2 = 10Q1^2 + 1RQ1^2 = a^2 + b^2 \)

\[ 10P1^2 = 10Q1^2 + 1PQ1^2 = a^2 + b^2 + c^2 \]

\( \Rightarrow 10P1 = \sqrt{a^2 + b^2 + c^2} \)
In the opposite direction, it is clear how to construct the point \( P \) of given coordinates \((a, b, c)\).

**Distance between two points** \( P_1(a_1, b_1, c_1) \) and \( P_2(a_2, b_2, c_2)\)

There is a unique point \( P \) in the 3D space s.t. \( O, P, P_2, P_1 \) are the consecutive vertices of a parallelogram.

The midpoints of \( OP_2 \) and \( PP_1 \) coincide, so

\[
\begin{align*}
\frac{0 + a_2}{2} &= \frac{x + a_1}{2} \\
0 + b_2 &= \frac{y + b_1}{2} \\
0 + c_2 &= \frac{z + c_1}{2}
\end{align*}
\]

\[\Rightarrow \begin{cases} x = a_2 - a_1 \\ y = b_2 - b_1 \Rightarrow P = P(a_2-a_1, b_2-b_1, c_2-c_1) \\ c = c_2 - c_1 \end{cases}\]

\[\Rightarrow |P_1P_2| = 10|P| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2}\]

**Corollary** The equation of the sphere of center \( C(a, b, c) \) and radius \( r \geq 0 \) is given by

\[(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\]

The points \( P(x, y, z) \) inside the closed ball of center \( C(a, b, c) \) and radius \( r \geq 0 \) are characterized by

\[(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2\]

**Vectors** From physics we know that vectors (in 2D or 3D) are characterized by **direction** and **magnitude**, except for the zero vector \( \vec{0} \) which has no direction but magnitude 0.

Two vectors are equivalent (equal) when they have the same direction and the same magnitude.
Given a vector \( \overrightarrow{P_1P_2} \) with tail \( P_1(a_1, b_1, c_1) \) and head \( P_2(a_2, b_2, c_2) \), there is a unique point \( P \) in the 3D space s.t. \( \overrightarrow{P_1P} = \overrightarrow{OP} \). We have seen that

\[
P = P(a_2-a_1, b_2-b_1, c_2-c_1).
\]

We will denote \( \overrightarrow{P_1P} = \overrightarrow{OP} = \langle a_2-a_1, b_2-b_1, c_2-c_1 \rangle \). Note that \( \overrightarrow{OP} = \overrightarrow{OP_2} - \overrightarrow{OP_1} \).

**Operations with Vectors**

1) **Addition**: defined using the Parallelogram Rule

\[
\vec{V} + \vec{W} \quad \text{on the Triangle Rule.}
\]

Easy to check that

\[
\begin{align*}
\vec{V} &= \langle x_1, y_1, z_1 \rangle \\
\vec{W} &= \langle x_2, y_2, z_2 \rangle
\end{align*}
\]

\[
\Rightarrow \quad \vec{V} + \vec{W} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle
\]

**Properties**: \( \vec{0} + \vec{V} = \vec{V} \); \( \vec{V} + \vec{W} = \vec{W} + \vec{V} \) (commutativity)

\[
(\vec{V}_2 + \vec{V}_2) + \vec{V}_3 = \vec{V}_1 + (\vec{V}_2 + \vec{V}_3) = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 \quad \text{(associativity)}
\]

2) **Multiplication by scalars**

\[
\lambda \in \mathbb{R} \quad \lambda \vec{V} \quad \text{is the vector}
\]

of magnitude \( |\lambda| |\vec{V}| \)

\[
|\lambda| = \sqrt{\lambda^2}
\]

\[
\lambda > 0 \quad \text{same direction as} \quad \vec{V}
\]

\[
\lambda < 0 \quad \text{opposite direction of} \quad \vec{V}
\]

\[
\overrightarrow{0} = \overrightarrow{1}\vec{V} = \vec{V}
\]

\[
\overrightarrow{-1} = \overrightarrow{(-1)}\vec{V} = \vec{V}
\]
Application: parameterization of lines and line segments

\[ \overrightarrow{OP} = \overrightarrow{OP_1} + t \overrightarrow{P_1 P_2} \text{ for some } t \in \mathbb{R} \]

In coordinates this reads as:

\[
\begin{align*}
x &= a_1 + t(b_1 - a_1) = (1-t)a_1 + ta_2 \\
y &= b_1 + t(b_2 - b_1) = (1-t)b_1 + tb_2 \\
z &= c_1 + t(c_2 - c_1) = (1-t)c_1 + tc_2
\end{align*}
\]

or

\[
\overrightarrow{OP} = (1-t)\overrightarrow{OP_1} + t\overrightarrow{OP_2} = (1-t)\overrightarrow{OP_1} + t\overrightarrow{OP_2} = (1-t)\overrightarrow{P_1} + t\overrightarrow{P_2}
\]

The closed segment \([P_1, P_2]\) is parameterized by the same equations but asking that \(t \in [0,1]\) instead of \(t \in \mathbb{R}\).

§ 12.3 Dot Product

Given \(\overrightarrow{a} = \langle a_1, b_1, c_1 \rangle\), \(\overrightarrow{b} = \langle a_2, b_2, c_2 \rangle\), define the dot product

\[ \overrightarrow{a} \cdot \overrightarrow{b} = a_1 a_2 + b_1 b_2 + c_1 c_2 \]

Real number

We will show that the famous CAUCHY-SCHWARZ inequality holds:

\[ |\overrightarrow{a} \cdot \overrightarrow{b}| \leq |\overrightarrow{a}| |\overrightarrow{b}| \]

\[ -1 \leq \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}| |\overrightarrow{b}|} \leq 1 \text{ whenever } \overrightarrow{a} \neq \overrightarrow{0}, \overrightarrow{b} \neq \overrightarrow{0}.
\]

\[ \Rightarrow \text{ there is a unique angle } \theta \in [0,\pi] \text{ such that } \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}| |\overrightarrow{b}|} = \cos \theta,
\]

or equivalently

\[ \overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}| |\overrightarrow{b}| \cos \theta \]

\(\theta\) is called the angle between the vectors \(\overrightarrow{a}\) and \(\overrightarrow{b}\).