ERGODIC ACTIONS OF COMPACT MATRIX PSEUDOGROUPS ON C^*-ALGEBRAS

FLORIN P. BOCA

Dedicated to Professor Masamichi Takesaki on the occasion of his 60th birthday

Let G be a compact group acting on a unital C^*-algebra $\mathscr{M}$. The action is said to be ergodic if the fixed point algebra $\mathscr{M}^G$ reduces to scalars. The first breakthrough in the study of such actions was the finiteness theorem of Høegh-Krohn, Landstad and Størmer [HLS]. They proved that the multiplicity of each $\pi \in \widehat{G}$ in $\mathscr{M}$ is at most $\dim(\pi)$ and the unique G-invariant state on $\mathscr{M}$ is necessarily a trace. When combined with Landstad's result [L] that finite dimensionality for the spectral subspaces of actions of compact groups implies that the crossed product is a type I C^*-algebra (and in fact, as pointed out in [Wa1] is a direct sum of algebras of compact operators), the finiteness theorem shows that the crossed product of a unital C^*-algebra by an ergodic action of a compact group is necessarily equal to $\bigoplus_i \mathscr{K}(\mathscr{H}_i)$.

The study of such actions was essentially pushed forward by Wassermann. He developed an outstanding machinery based on the notion of multiplicity maps, establishing a remarkable connection with the equivariant K-theory. This approach allowed him to prove, among other important things, the strong negative result that SU(2) cannot act ergodically on the hyperfinite II_1 factor [Wa3].

The aim of this note is to study ergodic actions of Woronowicz's compact matrix pseudogroups on unital C^*-algebras, extending some of the previous results. In this insight we prove in §1 the analogue of the finiteness theorem. More precisely, if $G = (A, u)$ is a compact matrix pseudogroup acting ergodically on a unital C^*-algebra by a coaction $\sigma : \mathscr{M} \to \mathscr{M} \otimes A$ such that $\sigma(\mathscr{M})(1_{\mathscr{M}} \otimes A)$ is dense on $\mathscr{M} \otimes A$, then there is a decomposition of the *-algebra of σ-finite elements into isotypic subspaces $\mathscr{M}_0 = \bigoplus_{\alpha \in \widehat{G}} \mathscr{M}_\alpha$, orthogonal with respect to the scalar product induced on $\mathscr{M}$ by the unique σ-invariant state ω. Moreover, the spectral subspaces $\mathscr{M}_\alpha$ are finite dimensional and $\dim(\mathscr{M}_\alpha) \leq M_\alpha^2$, M_α being the *quantum dimension* of $\alpha \in \widehat{G}$. If the Haar measure is faithful on A, then $\mathscr{M}_0$ is dense in the GNS Hilbert space $\mathscr{H}_\omega$.

Although ω is not in general a trace, we prove the existence of a multiplicative linear map $\Theta : \mathscr{M}_0 \to \mathscr{M}_0$ such that $\omega(xy) = \omega(\Theta(y)x)$ for all $x \in \mathscr{M}$, $y \in \mathscr{M}_0$ and Θ is a scalar multiple of the modular operator F_α when restricted to each irreducible σ-invariant subspace of the spectral subspace $\mathscr{M}_\alpha$.

The crossed products by such coactions are studied in §2 where we prove, using the Takesaki-Takai type duality theorem of Baaj and Skandalis [BS], that they are isomorphic all the time to a direct sum of C^*-algebras of compact operators. As a corollary, if a compact matrix pseudogroup

The paper has appeared in *Recent Advances in Operator Algebras* (Orléans, 1992), Astérisque No. 232 (1995), 93–109, available online at `http://www.numdam.org/article/AST_1995__232__93_0.pdf`. This version fixes misprints in the diagrams and some equation tags.

with underlying nuclear C^*-algebra acts ergodically on a unital C^*-algebra $\mathscr{M}$, then $\mathscr{M}$ follows nuclear.

We are grateful to Masamichi Takesaki for discussions on [BS] and [Wor], to Gabriel Nagy for valuable discussions on the paper and related topics and to the referee for his pertinent considerations. Our thanks go to Magnus Landstad for his remarks on a preliminary draft of this paper. After a discussion with him, I realized that the dimension of the spectral subspaces of α is bounded by M_α^2, improving a previous rougher estimation.

1. The isotypic decomposition and finiteness of multiplicities for ergodic actions

We start with a couple of definitions.

Definition 1 ([BS])**.** *A coaction of a unital Hopf C^*-algebra (A, Δ_A) on a unital C^*-algebra $\mathscr{M}$ is a unital one-to-one *-homomorphism $\sigma : \mathscr{M} \to \mathscr{M} \otimes A$ (the tensor product will be all the time the minimal C^*-one) that makes the following diagram commutative*

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\quad\sigma\quad} & \mathscr{M} \\ \sigma\big\downarrow & & \big\downarrow id_{\mathscr{M}}\otimes\Delta_A \\ \mathscr{M}\otimes A & \xrightarrow{\sigma\otimes id_A} & \mathscr{M}\otimes A\otimes A \end{array}$$

A C^*-algebra $\mathscr{M}$ with a coaction σ of (A, Δ_A) is called an A-algebra if σ is one-to-one and $\sigma(\mathscr{M})(1_{\mathscr{M}} \otimes A)$ is dense in $\mathscr{M} \otimes A$.

Definition 2. *The fixed points of the coaction $\sigma : \mathscr{M} \to \mathscr{M} \otimes A$ are the elements of $\mathscr{M}^\sigma = \{x \in \mathscr{M} \mid \sigma(x) = x \otimes 1_A\}$. The coaction σ is called ergodic if $\mathscr{M}^\sigma = \mathbb{C}1_{\mathscr{M}}$.*

We denote by $\mathscr{M}^*$ the set of continuous linear functionals on $\mathscr{M}$.

Definition 3. *$\phi \in \mathscr{M}^*$ is called σ-invariant if*

$$(\phi \otimes \psi)(\sigma(x)) = (\phi \otimes \psi)(x \otimes 1_A) = \psi(1_A)\phi(x) \quad \textit{for all } \psi \in A^*.$$

Let $G = (A, u)$ be a compact matrix pseudogroup with comultiplication $\Delta_A : A \to A \otimes A$, smooth structure $\mathscr{A}$ and coinverse $\kappa : \mathscr{A} \to \mathscr{A}$ (cf. [Wor]). Then A^* is an algebra with respect to the convolution $\phi * \psi = (\phi \otimes \psi)\Delta_A$, $\phi, \psi \in A^*$ and there exists a unique state h on A, called the Haar measure on G, so that $\phi * h = h * \phi = \phi(1_A)h$ for all $\phi \in A^*$. Let $\mathscr{M}$ be a unital C^*-algebra which is an A-algebra via the coaction $\sigma : \mathscr{M} \to \mathscr{M} \otimes A$ and consider $\theta = (id_{\mathscr{M}} \otimes h)\sigma$.

Lemma 4. i) *$\theta(x) \in \mathscr{M}^\sigma$ for all $x \in \mathscr{M}$. Moreover θ is a conditional expectation from $\mathscr{M}$ onto $\mathscr{M}^\sigma$.*

ii) *If σ is ergodic, then $\theta(x) = \omega(x)1_{\mathscr{M}}$ and ω is the only σ-invariant state on $\mathscr{M}$.*

Proof. i) Note that

$$\begin{aligned} \sigma(\theta(x)) &= \sigma((id_{\mathscr{M}} \otimes h)(\sigma(x)) = (\sigma \otimes h)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes id_A \otimes h)(\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes id_A \otimes h)(id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes (id_A \otimes h)\Delta_A)(\sigma(x)), \qquad x \in \mathscr{M}. \end{aligned}$$

Since $(id_A \otimes h)(\Delta_A(a)) = h * a = h(a)1_A$, $a \in A$ ([Wor, 4.2]), we obtain further:

$$\begin{aligned}(\psi_1 \otimes \psi_2)((id_{\mathscr{M}} \otimes (id_A \otimes h)\Delta_A)(y \otimes a)) &= (\psi_1 \otimes \psi_2)(y \otimes (id_A \otimes h)(\Delta_A(a))) \\ &= (\psi_1 \otimes \psi_2)(y \otimes h(a)1_A) = \psi_1(y)\psi_2(1_A)h(a) \\ &= (\psi_1 \otimes \psi_2)((id_{\mathscr{M}} \otimes h)(y \otimes a) \otimes 1_A),\end{aligned}$$

for all $y \in \mathscr{M}$, $a \in A$, $\psi_1 \in \mathscr{M}^*$, $\psi_2 \in A^*$. Therefore for $x \in \mathscr{M}$, $\psi_1 \in \mathscr{M}^*$, $\psi_2 \in A^*$ we have:

$$(\psi_1 \otimes \psi_2)(\sigma(\theta(x)) = (\psi_1 \otimes \psi_2)((id_{\mathscr{M}} \otimes h)\sigma(x) \otimes 1_A) = (\psi_1 \otimes \psi_2)(\theta(x) \otimes 1_A)$$

and consequently $\sigma(\theta(x)) = \theta(x) \otimes 1_A$ for all $x \in \mathscr{M}$. θ is a norm one projection since

$$\theta(x) = (id_{\mathscr{M}} \otimes h)(\sigma(x)) = (id_{\mathscr{M}} \otimes h)(x \otimes 1) = x, \qquad x \in \mathscr{M}^\sigma.$$

ii) Let $\psi \in A^*$. Then we get for all $x \in \mathscr{M}$:

$$\begin{aligned}(\omega \otimes \psi)(\sigma(x)) &= ((id_{\mathscr{M}} \otimes h)\sigma \otimes \psi)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes \psi)((id_{\mathscr{M}} \otimes h)\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes \psi)(id_M \otimes h \otimes id_A)((\sigma \otimes id_A)\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes \psi)(id_{\mathscr{M}} \otimes h \otimes id_A)((id_{\mathscr{M}} \otimes \Delta_A)\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes \psi)(id_{\mathscr{M}} \otimes (h \otimes id_A)\Delta_A)(\sigma(x)) \\ &= \psi(1_A)(id_{\mathscr{M}} \otimes h)(\sigma(x)) = \psi(1_A)\omega(x),\end{aligned}$$

therefore ω is a σ-invariant state on $\mathscr{M}$. Finally, assume that ϕ is a σ-invariant state on $\mathscr{M}$. Then for all $x \in \mathscr{M}$:

$$\phi(x) = (\phi \otimes h)(\sigma(x)) = \phi((id_{\mathscr{M}} \otimes h)(\sigma(x))) = \phi(\theta(x)) = \phi(\omega(x)1_{\mathscr{M}}) = \omega(x). \qquad \square$$

Remarks. 1) The proof of the previous statement doesn't use the faithfulness of σ but only the equality $(\sigma \otimes id_A)\sigma = (id_{\mathscr{M}} \otimes \Delta_A)\sigma$.

2) Since the tensor product of two faithful completely positive maps is still faithful [T], it follows that if σ is one-to-one and h is faithful on A, then ω is a faithful state on $\mathscr{M}$.

3) Although one can easily pass from a compact matrix pseudogroup to the reduced one, which has faithful Haar measure, as indicated at page 656 in [Wor], it turns out that the Haar measure is faithful in several important examples (e.g. on commutative CMP, on reduced cocommutative CMP or, cf. [N], on $\mathrm{SU}_\mu(N)$).

Denote by $\mathscr{H}_\omega$ the completion of $\mathscr{M}$ with respect to the inner product $\langle x, y\rangle_2 = \omega(y^*x)$ and let $\mathscr{M}$ acting on $\mathscr{H}_\omega$ in the GNS representation. Consider the C^*-Hilbert module $\mathscr{H}_\omega \otimes A$ with the A-valued inner product $\langle x_\omega \otimes a, y_\omega \otimes b\rangle_A = \omega(y^*x)b^*a$ for $x, y \in \mathscr{M}$, $a, b \in A$, which can be viewed as $\mathscr{M} \otimes A$ in the Stinespring representation of the completely positive map $\omega \otimes id_A$. Define also $V : \mathscr{H}_\omega \otimes A \to \mathscr{H}_\omega \otimes A$ by:

$$V\left(\sum_i (x_i)_\omega \otimes a_i\right) = \sum_i \sigma(x_i)(1_\omega \otimes a_i), \qquad x_i \in \mathscr{M}, a_i \in A.$$

Lemma 5. *V is a unitary in $\mathscr{L}(\mathscr{H}_\omega \otimes A) = M(\mathscr{K}(\mathscr{H}_\omega) \otimes A)$ (the multipliers of the C^*-algebra $\mathscr{K}(\mathscr{H}_\omega) \otimes A$) and $\sigma(x) = V(x \otimes 1_A)V^*$, $x \in \mathscr{M}$.*

Proof. For any $\phi \in A^*$, $a, b \in A$, denote by $\phi(a \cdot b) \in A^*$ the linear functional $\phi(a \cdot b)(x) = \phi(axb)$, $x \in A$. The σ-invariance of ω yields:

$$\phi((\omega \otimes id_A)((1_{\mathscr{M}} \otimes b^*)\sigma(x)(1_{\mathscr{M}} \otimes a)) = (\omega \otimes \phi(b^* \cdot a))(\sigma(x))$$
$$= \phi(b^* \cdot a)(1_A)\omega(x) = \phi(b^* a)\omega(x), \qquad x \in \mathscr{M}, a, b \in A, \phi \in A^*,$$

therefore we have for all $a, b \in A$, $x, y \in \mathscr{M}$:

$$\langle V(x_\omega \otimes a), V(y_\omega \otimes b)\rangle_A = \langle \sigma(x)(1_\omega \otimes a), \sigma(y)(1_\omega \otimes b)\rangle_A$$
$$= (\omega \otimes id_A)((1_{\mathscr{M}} \otimes b^*)\sigma(y^* x)(1_{\mathscr{M}} \otimes a)) = \omega(y^* x)b^* a = \langle x_\omega \otimes a, y_\omega \otimes b\rangle_A$$

and V follows isometry on $\mathscr{H}_\omega \otimes A$. Furthermore V is unitary since $\sigma(\mathscr{M})(1_{\mathscr{M}} \otimes A)$ is dense in $\mathscr{M} \otimes A$ and the relation $V(x \otimes 1_A) = \sigma(x)V$, $x \in \mathscr{M}$, is obvious. □

Definition 6 ([BS])**.** *A corepresentation of the Hopf C^*-algebra (A, Δ_A) is a unitary $V \in \mathscr{L}(\mathscr{H}_V \otimes A) = M(\mathscr{K}(\mathscr{H}_V) \otimes A)$ such that*

$$V_{12}V_{13} = (id_{\mathscr{L}(\mathscr{H}_V)} \otimes \Delta_A)(V).$$

All the corepresentations throughout this paper will be unitary unless specified otherwise. Note that in the case when $\dim \mathscr{H}_V < \infty$, V is called [Wor] a (finite dimensional) representation of the quantum matrix pseudogroup $G = (A, u)$. Thus the representations of the quantum matrix pseudogroup $G = (A, u)$ are the corepresentations of (A, Δ_A) and we will call them simply the corepresentations of A.

Lemma 7. *The unitary V from Lemma 5 is a corepresentation of A.*

Proof. Let $T = (id_{\mathscr{L}(\mathscr{H}_\omega)} \otimes \Delta_A)(V) \in \mathscr{L}(\mathscr{H}_\omega \otimes A \otimes A) = M(\mathscr{K}(H_\omega) \otimes A \otimes A)$. Since $1_\omega \otimes 1_A$ is fixed by V, then $\langle V(1_\omega \otimes 1_A), x_\omega^* \otimes 1_A\rangle_A = (\omega(x\,\cdot\,) \otimes id_A)(V) = \omega(x)1_A$ and consequently:

$$(\omega(x\,\cdot\,) \otimes id_{A\otimes A})(id_{\mathscr{L}(\mathscr{H}_\omega)} \otimes \Delta_A)(V) = (\omega(x\,\cdot\,) \otimes \Delta_A)(V)$$
$$= \Delta_A((\omega(x\,\cdot\,) \otimes id_A)(V) = \Delta_A(\omega(x)1_A) = \omega(x)1_{A\otimes A}, \qquad x \in \mathscr{M}.$$

Then, for any $a, a', b, b' \in A$:

$$\langle T(1_\omega \otimes a \otimes b), x_\omega \otimes a' \otimes b'\rangle_A = (\omega \otimes id_{A\otimes A})((x^* \otimes a'^* \otimes b'^*)T(1_{\mathscr{M}} \otimes a \otimes b))$$
$$= (a'^* \otimes b'^*)((\omega(x^*\,\cdot\,) \otimes id_{A\otimes A})(T))(a \otimes b)$$
$$= \omega(x^*)a'^* a \otimes b'^* b = \langle 1_\omega \otimes a \otimes b, x_\omega \otimes a' \otimes b'\rangle_A,$$

therefore $T(1_\omega \otimes a \otimes b) = 1_\omega \otimes a \otimes b$. Furthermore, since T is unitary, we also have $T^*(1_\omega \otimes a \otimes b) = 1_\omega \otimes a \otimes b$ and for any $x \in \mathscr{M}$, $a, b \in A$:

$$V_{12}V_{13}(x_\omega \otimes a \otimes b) = V_{12}(\sigma(x)_{13}(1_\omega \otimes a \otimes b)) = (\sigma \otimes id_A)(\sigma(x))(1_\omega \otimes a \otimes b)$$
$$= (id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x))(1_\omega \otimes a \otimes b)$$
$$= T(x \otimes 1_A \otimes 1_A)T^*(1_\omega \otimes a \otimes b) = T(x_\omega \otimes a \otimes b).$$ □

Remark 8. *Let $W \in \mathscr{L}(\mathscr{H}_W \otimes A)$ be a corepresentation, $\mathscr{H}_V \subset \mathscr{H}_W$ be a closed subspace of $\mathscr{H}_W$ and denote by P the orthogonal projection from $\mathscr{H}_W$ onto $\mathscr{H}_V$. If $(P \otimes id_A)W = W(P \otimes id_A)$, then $V = W(P \otimes id_A) \in \mathscr{L}(\mathscr{H}_V \otimes A)$ is by definition a subcorepresentation of W. Clearly $V^\perp = W(P^\perp \otimes id_A)$, where $P^\perp = I_{\mathscr{L}(\mathscr{H}_W)} - P$, is also a subcorepresentation of W.*

For $V \in M(\mathscr{K}(\mathscr{H}_V) \otimes A$ and $\rho \in A^*$ define as in [Wor] $V_\rho = (id_{\mathscr{L}(\mathscr{H}_V)} \otimes \rho)V \in \mathscr{L}(\mathscr{H}_V)$. Then Lemma 6 yields for any $\rho, \rho' \in A^*$:

$$\begin{aligned} V_\rho V_{\rho'} &= (id \otimes \rho)(V)(id \otimes \rho')(V) = (id \otimes \rho \otimes \rho')(V_{12}V_{13}) \\ &= (id \otimes \rho \otimes \rho')(id \otimes \Delta_A)(V) = (id \otimes (\rho * \rho'))(V) = V_{\rho*\rho'}. \end{aligned}$$

One checks immediately as in [Wor, 4.3] that $E = V_h$ is the projection of $\mathscr{H}_V$ onto the subspace $\{\xi \in \mathscr{H}_V \mid V_\rho \xi = \rho(1_A)\xi, \forall \rho \in A^*\}$ of all V-invariant vectors of $\mathscr{H}_V$.

The tensor product of the corepresentations $V \in \mathscr{L}(\mathscr{H}_V \otimes A)$ and $W \in \mathscr{L}(\mathscr{H}_W \otimes A)$ is defined as in the finite dimensional case by $V \odot W = V_{13}W_{23} \in \mathscr{L}(\mathscr{H}_V \otimes \mathscr{H}_W \otimes A)$ and is still a corepresentation since

$$\begin{aligned} (id \otimes \Delta_A)(V \odot W) &= (id \otimes \Delta_A)(V_{13})(id \otimes \Delta_A)(W_{23}) \\ &= (V_{13}W_{23})_{12}(V_{13}W_{23})_{13} = (V \odot W)_{12}(V \odot W)_{13}. \end{aligned}$$

When $\dim \mathscr{H}_W < \infty$, W^c denotes as in [Wor] its contragradient representation, acting on the conjugate Hilbert space $\mathscr{H}'_W$.

Lemma 9. *If the Haar measure is faithful on A and $V \in M(\mathscr{K}(\mathscr{H}_V) \otimes A)$ is a corepresentation of A such that $(V \odot \alpha^c)_h = 0$ for all $\alpha \in G^c$, then $V = 0$.*

Proof. Denote $d_\alpha = \dim(\alpha)$. Then $\alpha = \sum_{i,j=1}^{d_\alpha} e_{ij} \otimes u_{ij}^\alpha$, $u_{ij} \in \mathscr{A}$, where $\{e_{ij}\}_{1 \le i,j \le d_\alpha}$ is the matrix unit of $\mathscr{L}(\mathscr{H}_\alpha)$ and $\alpha^c = \sum_{i,j=1}^{d_\alpha} e_{ij}^T \otimes u_{ji}^{\alpha *} \in \mathscr{L}(\mathscr{H}'_\alpha) \otimes \mathscr{A}$. If ϕ_{ij} denotes the linear functional on $\mathscr{L}(\mathscr{H}'_\alpha)$ defined by $\phi_{ij}(e_{kl}^T) = \delta_{ik}\delta_{jl}$ we obtain for all $\phi \in \mathscr{L}(\mathscr{H}_V)_*$:

$$(\phi \otimes \phi_{ij} \otimes id_A)(id_{\mathscr{L}(\mathscr{H}_V \otimes \mathscr{H}'_\alpha)} \otimes h)(V \odot \alpha^c) = h((\phi \otimes id_A)(V)u_{ji}^{\alpha *}).$$

Since $\{u_{ij}^\alpha\}_{1 \le i,j \le d_\alpha, \alpha \in \widehat{G}}$ is a linear basis in $\mathscr{A}$, then $h((\phi \otimes id_A)(V)a) = 0$ for all $a \in \mathscr{A}$ and therefore for all $a \in A$. But h is faithful on A hence $(\phi \otimes id_A)(V) = 0$ for all $\phi \in \mathscr{L}(\mathscr{H}_V)$, and therefore $V = 0$. ☐

Corollary 10. i) *If the Haar measure is faithful on A, then there are no irreducible infinite dimensional corepresentations of A.*

ii) *All the finite dimensional corepresentations (not necessarily unitary) of A are smooth (compare with a related question at page* 636 *in* [Wor]*).*

Proof. i) Let V be an infinite dimensional corepresentation of A on $\mathscr{H}_V$. By the previous Lemma, there exists $\alpha \in \widehat{G}$ such that $(V \odot \alpha^c)_h \neq 0$. Pick a nonzero element $S \in \operatorname{Ran}(V \otimes \alpha^c)_h$. Since α is finite dimensional, $S \in \operatorname{Mor}(\alpha, V)$. But $\operatorname{Ker} S$ is α-invariant and α is irreducible, thus S is one-to-one and we get the V-invariant finite dimensional subspace $\operatorname{Ran} S \subset \mathscr{H}_V$, contradicting the irreducibility of V.

ii) Consider now a finite dimensional corepresentation V of A, not necessary unitary. The statement in the previous lemma holds true, thus either V is completely degenerate or there exists $\alpha \in \widehat{G}$ such that $(V \odot \alpha^c)_h \neq 0$ and $0 \neq S \in \operatorname{Mor}(\alpha, V)$. It follows that $\mathscr{H}_V$ contains a V-invariant subspace $\mathscr{K} = \operatorname{Ran} S$ and $V|_{\mathscr{K}}$ is equivalent to α, therefore it is smooth, irreducible and nondegenerate. By [Wor, Prop.4.6] $\mathscr{K}$ has a V-invariant complement and the process continues until we write $\mathscr{H}_V$ as a direct sum of linear subspaces $\mathscr{H}_V = \bigoplus_i \mathscr{H}_i \bigoplus \mathscr{H}_0$, each subspace being V-invariant, $V|_{\mathscr{H}_i}$ being equivalent to a corepresentation from $\widehat{G}$ and $V|_{\mathscr{H}_0}$ completely nondegenerate. $\square$

The statements in the previous corollary are implicit in S. L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups, *Invent. Math. 93(1988), 35-76*, as the referee informed us.

For any $\alpha \in \widehat{G}$ consider as in [Wor, §5] $\rho_\alpha \in A^*$, $\rho_\alpha(a) = M_\alpha h((f_1 * \chi_\alpha)^* a)$. Then $\rho_\alpha * \rho_\beta = \delta_{\alpha\beta}\rho_\alpha$ for all $\alpha, \beta \in \widehat{G}$. Therefore, if $U \in M(\mathscr{K}(\mathscr{H}_U) \otimes A)$ is a representation of A, $P_\alpha = U_{\rho_\alpha} = (id_{\mathscr{L}(\mathscr{H}_U)} \otimes \rho_\alpha)(U)$ are mutually orthogonal projections, but they are not self-adjoint in general. Consider also the bounded linear maps $P_\alpha : \mathscr{M} \to \mathscr{M}$, $P_\alpha(x) = (id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))$. Then we have for all $x \in \mathscr{M}$:

$$\begin{aligned} P_\alpha P_\beta(x) &= (id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma((id_{\mathscr{M}} \otimes \rho_\beta)(\sigma(x))) \\ &= (id_{\mathscr{M}} \otimes \rho_\alpha)((id_{\mathscr{M}\otimes A} \otimes \rho_\beta)(\sigma \otimes id_A)(\sigma(x))) \\ &= (id_{\mathscr{M}} \otimes \rho_\alpha)((id_{\mathscr{M}\otimes A} \otimes \rho_\beta)(id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x))) \\ &= (id_{\mathscr{M}} \otimes \rho_\alpha(id_A \otimes \rho_\beta)\Delta_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes (\rho_\alpha * \rho_\beta))(\sigma(x)) = \delta_{\alpha\beta} P_\alpha(x). \end{aligned}$$

We denote by $\mathscr{M}_\alpha$ the closed subspace $P_\alpha(\mathscr{M})$ of $\mathscr{M}$, $\alpha \in \widehat{G}$ and call it the *spectral subspace* associated with α.

The following lemma contains a couple of properties of the functionals $\rho_\alpha \in A^*$, $\alpha \in \widehat{G}$.

Lemma 11. i) $\mathscr{M}_{\alpha^c} = \mathscr{M}_\alpha^*$ *for all* $\alpha \in \widehat{G}$.
ii) $(h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(a)_{12}\Delta_A(b)_{13}) = 0$ *for all* $a, b \in A$, $\alpha \neq \beta$ *in* $\widehat{G}$.

Proof. i) Let $x \in \mathscr{M}_\alpha$. Then $x = (id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))$ and we have to check that $x^* = (id_{\mathscr{M}} \otimes \rho_{\alpha^c})(\sigma(x^*))$. Obvious computations which are implicit in [Wor, 5.6] show that $(f_1 * \chi_{\alpha^c})^* = \chi_\alpha * f_{-1}$ and $M_\alpha = M_{\alpha^c}$, where $M_\alpha = f_{-1}(\chi_\alpha) = f_1(\chi_\alpha)$, therefore we have for any $a \in A$:

$$\begin{aligned} \overline{\rho_\alpha(a)} &= M_\alpha \overline{h((f_1 * \chi_\alpha)^* a)} = M_\alpha h(a^*(f_1 * \chi_\alpha)) \\ &= M_\alpha h(a^*(f_1 * \chi_\alpha * f_{-1} * f_1)) = M_\alpha h((\chi_\alpha * f_{-1})a^*) \\ &= M_\alpha h((f_1 * \chi_{\alpha^c})^* a^*) = M_{\alpha^c} h((f_1 * \chi_{\alpha^c})^* a^*) = \rho_{\alpha^c}(a^*). \end{aligned}$$

This shows that $x^* = (id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))^* = (id_{\mathscr{M}} \otimes \rho_{\alpha^c})(\sigma(x)^*) = (id_{\mathscr{M}} \otimes \rho_{\alpha^c})(\sigma(x^*))$.

ii) Since $\text{span}\{u^\gamma_{ij}\}_{1\le i,j\le d_\gamma, \gamma\in\widehat{G}}$ is norm dense in A it is enough to take $a = u^\gamma_{ij}$, $b = u^\theta_{kl}$ for some $\gamma, \theta \in \widehat{G}$ and to remark that:

$$\begin{aligned} (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(u^\gamma_{ij})_{12}\Delta_A(u^\theta_{kl})_{13}) &= (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)\Bigg(\sum_{r=1}^{d_\gamma}\sum_{s=1}^{d_\theta} u^\gamma_{ir}u^\theta_{ks} \otimes u^\gamma_{rj} \otimes u^\theta_{sl}\Bigg) \\ &= \sum_{r=1}^{d_\gamma}\sum_{s=1}^{d_\theta} h(u^\gamma_{ir}u^\theta_{ks})\delta_{\beta^c\gamma}\delta_{rj}\delta_{\alpha\theta}\delta_{sl} = h(u^{\beta^c}_{ij}u^\alpha_{kl}) = h(u^{\beta\,*}_{ij}u^\alpha_{kl}) = 0. \end{aligned}$$

$\square$

Corollary 12. $\omega(P_\beta(y)^* P_\alpha(x)) = \omega(P_\alpha(x)P_\beta(y)^*) = 0$ *for all* $x, y \in \mathscr{M}$, $\alpha \neq \beta$ *in* $\widehat{G}$.

Proof. Denote $x_0 = P_\alpha(x)$, $y_0 = P_\beta(y)$. By the previous lemma $x_0^* \in \operatorname{Ran} P_{\alpha^c}$ and $y_0^* \in \operatorname{Ran} P_{\beta^c}$. Then we obtain:

$$\begin{aligned}\sigma(x_0) &= \sigma((id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))) = (id_{\mathscr{M}\otimes A} \otimes \rho_\alpha)(id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x))\end{aligned}$$

and the similar relations for the pairs (x_0^*, α^c), (y_0, β), (y_0^*, β^c). The σ-invariance of ω and the previous relations yield:

$$\begin{aligned}\omega(y_0^* x_0) &= (\omega \otimes h)(\sigma(y_0^*)\sigma(x_0)) \\ &= (\omega \otimes h)((id_{\mathscr{M}} \otimes (id_A \otimes \rho_{\beta^c})\Delta_A)(\sigma(y^*))(id_{\mathscr{M}} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x))),\end{aligned}$$

which is equal to 0 by Lemma 11 ii) since we have for any $a, b \in A$:

$$h((id_A \otimes \rho_{\beta^c})(\Delta_A(a))(id_A \otimes \rho_\alpha)(\Delta_A(b)) = (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(a)_{12}\Delta_A(b)_{13}) = 0.$$

The equality $\omega(x_0 y_0^*) = 0$ follows similarly. □

The previous corollary actually shows that the spectral subspaces $\mathscr{M}_\alpha$, $\alpha \in \widehat{G}$ are mutually orthogonal with respect to both scalar products $\langle x, y\rangle_{2,\omega} = \omega(y^* x)$ and $\langle x, y\rangle_{1,\omega} = \omega(x y^*)$ on $\mathscr{M}$.

The next statement is the analogue of the decomposition of a representation of a compact Lie group into isotypic subrepresentations.

Proposition 13. i) *The spectral subspaces $\mathscr{M}_\alpha$, $\alpha \in \widehat{G}$ are σ-invariant. Moreover*

$$\sigma(\mathscr{M}_\alpha) \subset \mathscr{M}_\alpha \otimes \mathscr{A}_\alpha = \operatorname{span}\{x_{ij} \otimes u_{ij}^\alpha \mid x_{ij} \in \mathscr{M}_\alpha,\ 1 \le i, j \le d_\alpha\}$$

and if V is a finite dimensional σ-invariant subspace of $\mathscr{M}_\alpha$, then α is the only irreducible subcorepresentation of $\sigma|_V$.

ii) *Any finite dimensional subspace of $\mathscr{M}_\alpha$ is contained in a finite dimensional σ-invariant subspace of $\mathscr{M}_\alpha$.*

iii) *If the Haar measure is faithful on A, then $\mathscr{M}_0 = \operatorname{span}\{\mathscr{M}_\alpha \mid \alpha \in \widehat{G}\}$ is dense in the Hilbert space $\mathscr{H}_\omega$.*

Proof. i) Remark first that

$$\begin{aligned}\sigma(P_\alpha(x)) &= \sigma((id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))) = (id_{\mathscr{M}\otimes A} \otimes \rho_\alpha)(id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x)), \qquad x \in \mathscr{M}.\end{aligned}$$

Since $(id_A \otimes \rho_\alpha)(\Delta_A(A)) \subset \mathscr{A}_\alpha$ it follows that $\sigma(\mathscr{M}_\alpha) \subset \mathscr{M} \otimes \mathscr{A}_\alpha$. Using again the density of $\operatorname{span}\{\mathscr{A}_\beta \mid \beta \in \widehat{G}\}$ in A and the equality

$$(\rho_\alpha \otimes id_A)(\Delta_A(u_{ij}^\beta)) = \delta_{\alpha\beta} u_{ij}^\beta = (id_A \otimes \rho_\alpha)(\Delta_A(u_{ij}^\beta)),$$

we obtain $(\rho_\alpha \otimes id_A)\Delta_A = (id_A \otimes \rho_\alpha)\Delta_A$ and furthermore

$$\begin{aligned}(P_\alpha \otimes id_A)(\sigma(x)) &= ((id_{\mathscr{M}} \otimes \rho_\alpha)\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes \rho_\alpha \otimes id_A)(\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes \rho_\alpha \otimes id_A)(id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathscr{M}} \otimes (\rho_\alpha \otimes id_A)\Delta_A)(\sigma(x)) = \sigma(P_\alpha(x)), \qquad x \in \mathscr{M}.\end{aligned}$$

This shows in particular that $\sigma(\mathscr{M}_\alpha) \subset \mathscr{M}_\alpha \otimes \mathscr{A}_\alpha$. If $V \subset \mathscr{M}_\alpha$ is finite dimensional and σ-invariant, then $\sigma|_V$ contains only copies of α since $(id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x)) = x$, $x \in V$.

ii) It is enough to prove that for any $x \in \mathscr{M}_\alpha$, there exists $V_x \subset \mathscr{M}_\alpha$ finite dimensional σ-invariant space that contains x. Set $V_x = \{(id_{\mathscr{M}} \otimes \rho)(\sigma(x)) \mid \rho \in A^*\}$, subspace of $\mathscr{M}_\alpha$ which contains x since $x = (id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))$, and is finite dimensional since $\sigma(x) = \sum_{i,j=1}^{d_\alpha} x_{ij} \otimes u_{ij}^\alpha$ for some $x_{ij} \in \mathscr{M}_\alpha$. Therefore $V_x \subset \operatorname{span}\{x_{ij} \mid 1 \le i,j \le d_\alpha\}$ and finally V_x is σ-invariant since for any $\rho, \rho' \in A^*$, $x \in \mathscr{M}$ we have:

$$\begin{aligned}(id_{\mathscr{M}} \otimes \rho')(\sigma((id_{\mathscr{M}} \otimes \rho)(\sigma(x))) &= (id_{\mathscr{M}} \otimes \rho')((id_{\mathscr{M}\otimes A} \otimes \rho)(id_{\mathscr{M}} \otimes \Delta_A)(\sigma(x))) \\ &= (id_{\mathscr{M}} \otimes \rho'((id_A \otimes \rho)\Delta_A))(\sigma(x)) = (id_{\mathscr{M}} \otimes (\rho' * \rho))(\sigma(x)).\end{aligned}$$

iii) One can easily check as in the case when both corepresentations are finite dimensional that for any $\alpha \in \widehat{G}$ and any corepresentation $V \in \mathscr{L}(\mathscr{H}_V \otimes A)$ of A we still have:

$$\begin{aligned}\operatorname{Mor}(\alpha, V) &= \{\xi \in \mathscr{H}_V \otimes \mathscr{H}_\alpha' = \mathscr{L}(\mathscr{H}_\alpha, \mathscr{H}_V) \mid (V \odot \alpha^c)_\rho \xi = \rho(1_A)\xi,\ \forall \rho \in A^*\} \\ &= (V \odot \alpha^c)_h(\mathscr{H}_V \odot \mathscr{H}_\alpha').\end{aligned}$$

Assume now that $\overline{\mathscr{M}_0} \neq \mathscr{H}_\omega$. Then, since $U|_{\overline{\mathscr{M}_0}}$ is a subcorepresentation of U it follows that there exists $V \in \mathscr{L}(\mathscr{H}_V \otimes A)$ corepresentation of A with $0 \neq \mathscr{H}_V \subset \mathscr{H}_\omega \ominus \overline{\mathscr{M}_0}$. By Lemma 9 there exists $\beta \in \widehat{G}$ such that $(V \odot \beta^c)_h \neq 0$ and we find a nonzero $S \in \mathscr{L}(\mathscr{H}_\beta, \mathscr{H}_V)$ with $(S \otimes id_A)\beta = V(S \otimes id_A)$. Since $\operatorname{Ker} S$ is β-invariant and β irreducible, S follows one-to-one. But $\operatorname{Ran} S$ is a finite dimensional V-invariant subspace of $\mathscr{H}_V$, thus $\beta \preceq V$. Since $\mathscr{H}_V$ is orthogonal to $\mathscr{M}_0$ we have $(id_{\mathscr{L}(\mathscr{H}_\beta)} \otimes \rho_\alpha)(\beta) = 0$ for all $\alpha \in \widehat{G}$, therefore $\beta = 0$ by [Wor, 5.8]. But σ is one-to-one, thus V is nondegenerate and we get a contradiction. Consequently $\overline{\mathscr{M}_0} = \mathscr{H}_\omega$. □

Proposition 14. *$\mathscr{M}_0$ is a *-algebra and $\mathscr{M}_\alpha \mathscr{M}_\beta \subset \operatorname{span}\{\mathscr{M}_\gamma \mid \gamma \preceq \alpha \odot \beta\}$ for all $\alpha, \beta \in \widehat{G}$.*

Proof. Fix $\alpha, \beta \in \widehat{G}$, $u^\alpha \in \alpha$, $u^\beta \in \beta$ and $\mathscr{H}_\alpha \subset \mathscr{M}_\alpha$, $\mathscr{H}_\beta \subset \mathscr{M}_\beta$ irreducible finite dimensional σ-invariant subspaces. Let $\{e_i\}_{1 \le i \le d_\alpha}$ and $\{f_r\}_{1 \le r \le d_\beta}$ be orthonormal basis in $\mathscr{H}_\alpha$ and respectively $\mathscr{H}_\beta$ such that $\sigma(e_i) = \sum_{j=1}^{d_\alpha} e_j \otimes u_{ji}^\alpha$ and $\sigma(f_r) = \sum_{s=1}^{d_\beta} f_s \otimes u_{sr}^\beta$. The restriction of σ to $\mathscr{H}_\alpha$ (respectively to $\mathscr{H}_\beta$), $\widehat{V}_\alpha : \mathscr{H}_\alpha \to \mathscr{H}_\alpha \otimes A$ (respectively, $\widehat{V}_\beta : \mathscr{H}_\beta \to \mathscr{H}_\beta \otimes A$) implements u^α (respectively u^β). Then $\widehat{V} : \mathscr{H}_\alpha \otimes \mathscr{H}_\beta \to \mathscr{H}_\alpha \otimes \mathscr{H}_\beta \otimes A$, $\widehat{V}(x \otimes y) = \widehat{V}_\alpha(x)_{13} \widehat{V}_\beta(y)_{23} = \sigma(x)_{13}\sigma(y)_{23}$ implements $u^\alpha \odot u^\beta$. Consider the onto linear operator $S : \mathscr{H}_\alpha \otimes \mathscr{H}_\beta \to \operatorname{span} \mathscr{H}_\alpha \mathscr{H}_\beta$, $S(x \otimes y) = xy$. Since:

$$\begin{aligned}\widehat{V}(e_i \otimes f_r) &= \sum_{j=1}^{d_\alpha} \sum_{s=1}^{d_\beta} e_j f_s \otimes u_{ji}^\alpha u_{sr}^\beta = \left(\sum_{j=1}^{d_\alpha} e_j \otimes u_{ji}^\alpha \right) \left(\sum_{s=1}^{d_\beta} e_s \otimes u_{sr}^\beta \right) \\ &= \sigma(e_i)\sigma(f_r) = \sigma(e_i f_r) = \sigma S(e_i \otimes f_r),\end{aligned}$$

the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{H}_\alpha \otimes \mathscr{H}_\beta & \xrightarrow{\widehat{V}} & \mathscr{H}_\alpha \otimes \mathscr{H}_\beta \otimes A \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S\otimes id_A} \\ \operatorname{span} \mathscr{H}_\alpha\mathscr{H}_\beta & \xrightarrow{\sigma} & \operatorname{span} \mathscr{H}_\alpha\mathscr{H}_\beta \otimes A \end{array}$$

Therefore $S \in \operatorname{Mor}(u^\alpha \odot u^\beta, \sigma|_{\operatorname{span} \mathscr{H}_\alpha\mathscr{H}_\beta})$ and any irreducible subcorepresentation of $\sigma|_{\operatorname{span} \mathscr{H}_\alpha\mathscr{H}_\beta}$ should appear in $\alpha \odot \beta$, hence decomposing the last corepresentation into irreducible components we get $\mathscr{M}_\alpha\mathscr{M}_\beta \subset \operatorname{span}\{\mathscr{M}_\gamma \mid \gamma \preceq \alpha \odot \beta\}$. $\mathscr{M}_0$ is *-closed since $\mathscr{M}_{\alpha^c} = \mathscr{M}_\alpha^*$ by Lemma 11. □

Denote by $\mathscr{H}_h$ the completion of A with respect to the scalar product $\langle a, b\rangle_{2,h} = h(b^*a)$, $a, b \in A$ and consider $W(x_\omega \otimes a_h) = (\sigma(x)(1_{\mathscr{M}} \otimes a))(1_\omega \otimes 1_h)$, $x \in \mathscr{M}$, $a \in A$. Clearly $W \in \mathscr{L}(\mathscr{H}_\omega \otimes \mathscr{H}_h)$ is a unitary. Since $\rho_\alpha = M_\alpha h((f_1 * \chi_\alpha)^* \cdot)$ and $f_1 * \chi_\alpha \in A$, it follows that $\rho_\alpha \in \mathscr{L}(\mathscr{H}_h)_*$ and ρ_α coincides on $\mathscr{L}(\mathscr{H}_h)$ with the vector form $M_\alpha\langle \cdot 1_h, (f_1 * \chi_\alpha)_h\rangle_{2,h}$. Therefore it makes sense to consider $p(\alpha) = (id_{\mathscr{L}(\mathscr{H}_\omega)} \otimes \rho_\alpha)(W) \in \mathscr{L}(\mathscr{H}_\omega)$. A straightforward computation yields for any $x, y \in \mathscr{M}$:

$$\begin{aligned}
\langle P_\alpha(x)_\omega, y_\omega\rangle_2 &= \omega(y^* P_\alpha(x)) = \omega(y^*(id_{\mathscr{M}} \otimes \rho_\alpha)(\sigma(x))) \\
&= \omega((id_{\mathscr{M}} \otimes \rho_\alpha)((y^* \otimes 1_A)\sigma(x))) \\
&= (\omega \otimes \rho_\alpha)((y^* \otimes 1_A)\sigma(x)) = (\omega(y^* \cdot) \otimes h(M_\alpha(f_1 * \chi_\alpha)^* \cdot))\sigma(x) \\
&= \langle \sigma(x)(1_\omega \otimes 1_h), y_\omega \otimes M_\alpha((f_1 * \chi_\alpha)^*)_h\rangle_{2,\omega\otimes h} \\
&= \langle W(x_\omega \otimes 1_h), y_\omega \otimes M_\alpha((f_1 * \chi_\alpha)^*)_h\rangle_{2,\omega\otimes h} \\
&= \langle (id_{\mathscr{L}(\mathscr{H}_\omega)} \otimes \rho_\alpha)(W)x_\omega, y_\omega\rangle_2 = \langle p(\alpha)x_\omega, y_\omega\rangle_2.
\end{aligned}$$

We obtain:

Remark 15. *P_α extends to the bounded operator $p(\alpha) \in \mathscr{L}(\mathscr{H}_\omega)$ and $p(\alpha)$ is a projection from $\mathscr{H}_\omega$ onto $\mathscr{H}_\alpha = \{x_\omega \mid x \in \mathscr{M}_\alpha\}$. Moreover, it is easy to see that $p(\alpha)$ is self-adjoint.*

Lemma 16. $\displaystyle\sum_{i=1}^{d_\alpha} u_{pi}^{\alpha\,*}\kappa^2(u_{qi}^\alpha) = \delta_{pq}1_A = \sum_{i=1}^{d_\alpha} \kappa^2(u_{ip}^\alpha)u_{iq}^{\alpha\,*}$ *for all $\alpha \in \widehat{G}$, $1 \le p, q \le d_\alpha$.*

Proof. Using the antimultiplicativity of κ and $\kappa(u_{ij}^\alpha) = u_{ji}^{\alpha\,*}$ we obtain:

$$\sum_{i=1}^{d_\alpha} u_{pi}^{\alpha\,*}\kappa^2(u_{qi}^\alpha) = \sum_{i=1}^{d_\alpha} \kappa(u_{ip}^\alpha)\kappa^2(u_{qi}^\alpha) = \sum_{i=1}^{d_\alpha} \kappa(\kappa(u_{qi}^\alpha)u_{ip}^\alpha)) = \kappa\left(\sum_{i=1}^{d_\alpha} \kappa(u_{qi}^\alpha)u_{ip}^\alpha\right) = \delta_{pq}1_A,$$

$$\sum_{i=1}^{d_\alpha} \kappa^2(u_{ip}^\alpha)u_{iq}^{\alpha\,*} = \sum_{i=1}^{d_\alpha} \kappa(u_{qi}^\alpha\kappa(u_{ip}^\alpha)) = \delta_{pq}1_A.$$ □

Theorem 17. *If $G = (A, u)$ is a compact matrix pseudogroup and $\mathscr{M}$ is a unital C^*-algebra with an A-algebra structure given by the ergodic coaction $\sigma : \mathscr{M} \to \mathscr{M} \otimes A$, then the spectral subspaces $\mathscr{M}_\alpha$ are finite dimensional and* $\dim(\mathscr{M}_\alpha) \leqslant M_\alpha^2$.

Proof. Let $\alpha \in \widehat{G}$ and fix $u^\alpha \in \alpha$ acting on $\mathscr{H}_\alpha$ and an orthonormal basis $\xi_1, \dots, \xi_{d_\alpha}$ in $\mathscr{H}_\alpha$ such that $u^\alpha(\xi_i) = \sum_{r=1}^{d_\alpha} \xi_r \otimes u^\alpha_{ri}$. Let $V_1, \dots, V_N$ be mutually $\langle\ ,\ \rangle_{2,\omega}$ orthogonal irreducible σ-invariant sunbspaces and let $U_k \in \mathscr{L}(\mathscr{H}_\alpha, V_k)$, $1 \le k \le N$ be unitaries such that $\sigma U_k = (U_k \otimes id_A)\alpha$. Denoting $e_i^{(k)} = U_k \xi_i$ it follows that $e_1^{(k)}, \dots, e_{d_\alpha}^{(k)}$ is an orthonormal basis in V_k and

$$\sigma(e_i^{(k)}) = \sum_{r=1}^{d_\alpha} e_r^{(k)} \otimes u^\alpha_{ri}, \tag{1}$$

thus

$$\sigma\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)} e_i^{(l)\,*}\Big) = \sum_{i,r,s=1}^{d_\alpha} e_r^{(k)} e_s^{(l)\,*} \otimes u^\alpha_{ri} u^{\alpha\,*}_{si} = \sum_{r=1}^{d_\alpha} e_r^{(k)} e_r^{(l)\,*} \otimes 1_A$$

and $\sum_{i=1}^{d_\alpha} e_i^{(k)} e_i^{(l)\,*} \in \mathbb{C}1_{\mathscr{M}}$.

Moreover, the equality $(F_\alpha \otimes id_A)\alpha = \alpha^{cc} F_\alpha$ and the previous Lemma yield:

$$\begin{aligned}
\sigma\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)\,*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)})\Big) &= \sigma\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)\,*} U_l F_\alpha^{-1}(\xi_i)\Big) \\
&= \sum_{i=1}^{d_\alpha} \sigma(e_i^{(k)\,*})(U_l F_\alpha^{-1} U_l^* \otimes id_A)\alpha^{cc}(\xi_i) \\
&= \sum_{i=1}^{d_\alpha} \sigma(e_i^{(k)\,*}) \sum_{s=1}^{d_\alpha} U_l F_\alpha^{-1} U_l^*(e_s^{(l)}) \otimes \kappa^2(u^\alpha_{si}) \\
&= \sum_{r,s=1}^{d_\alpha} e_r^{(k)\,*} U_l F_\alpha^{-1} U_l^*(e_s^{(l)}) \otimes \sum_{i=1}^{d_\alpha} u^{\alpha\,*}_{ri} \kappa^2(u^\alpha_{si}) \\
&= \sum_{r=1}^{d_\alpha} e_r^{(k)\,*} U_l F_\alpha^{-1} U_l^*(e_r^{(l)}) \otimes 1_A,
\end{aligned}$$

thus

$$\begin{aligned}
\sum_{i=1}^{d_\alpha} e_i^{(k)\,*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)}) &= \omega\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)\,*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)})\Big) 1_{\mathscr{M}} \\
&= \sum_{i=1}^{d_\alpha} \langle U_l F_\alpha^{-1} U_l^*(e_i^{(l)}), e_i^{(k)} \rangle_{2,\omega} 1_{\mathscr{M}} = \delta_{kl} \operatorname{Tr}(F_\alpha^{-1}) 1_{\mathscr{M}} = \delta_{kl} M_\alpha 1_{\mathscr{M}}.
\end{aligned} \tag{2}$$

Since the modular operator F_α associated with u^α is positive [Wor, 5.4], the matrix $C = (\langle F_\alpha^{-1}\xi_i, \xi_j\rangle)_{ij} \in \operatorname{Mat}_{d_\alpha}(\mathbb{C})$ is positive definite.

Denote $C^{1/2} = (\lambda_{ij})_{ij}$, $x_{rl} = \sum_{j=1}^{d_\alpha} \lambda_{rj} e_j^{(l)}$ and $X = (x_{rl})_{rl} \in \operatorname{Mat}_{d_\alpha, N}(\mathscr{M})$. Then

$$\sum_{i=1}^{d_\alpha} e_i^{(k)\,*} U_l F_\alpha^{-1} \xi_i = \sum_{i,j=1}^{d_\alpha} \langle F_\alpha^{-1}\xi_i, \xi_j\rangle e_i^{(k)\,*} e_j^{(l)} = \sum_{r,i,j=1}^{d_\alpha} \overline{\lambda_{ri}} \lambda_{rj} e_i^{(k)\,*} e_j^{(l)} = \sum_{r=1}^{d_\alpha} x_{rk}^* x_{rl},$$

or in other words $X^*X = M_\alpha I$ in $\mathscr{M} \otimes \operatorname{Mat}_N(\mathbb{C})$. This yields $XX^* \le M_\alpha I$ in $\mathscr{M} \otimes \operatorname{Mat}_{d_\alpha}(\mathbb{C})$.

Consider the positive linear functional $\phi \in \mathrm{Mat}_{d_\alpha}(\mathbb{C})_*$, $\phi(Y) = \mathrm{Tr}(C^{-1}Y)$, Tr being the normalized trace on $\mathrm{Mat}_{d_\alpha}(\mathbb{C})$. Then

$$\|\phi\| = \phi(I) = \mathrm{Tr}(C^{-1}) = \sum_{i=1}^{d_\alpha}(C^{-1})_{ii} = \sum_{i=1}^{d_\alpha}\langle F_\alpha\xi_i, \xi_i\rangle = \mathrm{Tr}(F_\alpha) = M_\alpha$$

and the last inequality yields:

$$\begin{aligned} M_\alpha^2 \geq \|\omega\otimes\phi\|\cdot\|XX^*\| \geq (\omega\otimes\phi)(XX^*) &= \sum_{i,j=1}^{d_\alpha}\sum_{k=1}^{N}\phi(e_{ij})\omega(x_{ik}x_{jk}^*) \\ = \sum_{k=1}^{N}\sum_{i,j,r,s=1}^{d_\alpha}\lambda_{sj}(C^{-1})_{ji}\lambda_{ir}\omega(e_r^{(k)}e_s^{(k)*}) &= \sum_{k=1}^{N}\sum_{r=1}^{d_\alpha}\omega(e_r^{(k)}e_r^{(k)*}) \\ = \sum_{k=1}^{N}\sum_{r=1}^{d_\alpha}\|e_r^{(k)}\|_{1,\omega}^2. \end{aligned}$$

This inequality plays a crucial role since it shows that if $W_\alpha \subset \mathscr{M}_\alpha$ is a σ-invariant subspace and A_{W_α} is the positive invertible operator on W_α (with respect to $\langle\ ,\ \rangle_{2,\omega}$) such that $\langle A_{W_\alpha}(x), y\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega}$, $x, y \in W_\alpha$, then

$$\mathrm{Tr}(A_{W_\alpha}) \leq M_\alpha^2. \tag{3}$$

Consider now a finite dimensional σ-invariant subspace $W \subset \mathscr{M}_0$ and let A_W be the unique positive invertible operator on W with $\langle A_W(x), y\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega}$, $x, y \in W$. Since $\langle \mathscr{M}_\alpha, \mathscr{M}_\beta\rangle_{1,\omega} = 0$ for $\alpha \neq \beta$, it follows that A_W leaves invariant each spectral subspace W_α of W and

$$\mathrm{Tr}(A_W) = \sum_{\alpha\prec W}\mathrm{Tr}(A_{W_\alpha}) \leq \sum_{\sigma\prec W} M_\alpha^2. \tag{4}$$

If in addition A_W is *-invariant and $Jx = x^*$, $x \in \mathscr{M}$, then

$$\langle JA_W^{-1}Jx, y\rangle_{2,\omega} = \langle y^*, A_W^{-1}(x^*)\rangle_{1,\omega} = \langle y^*, x^*\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega} = \langle A_W(x), y\rangle_{2,\omega}, \qquad x, y \in W.$$

Therefore $JA_W^{-1}J = A_W$ and in particular if λ is an eigenvalue of A_W with multiplicity m_λ, so is λ^{-1} with the same multiplicity. It follows that

$$\mathrm{Tr}(A_W) = \sum_\lambda m_\lambda\lambda = m_1 + \sum_{\lambda>1} m_\lambda(\lambda+\lambda^{-1}) \geq m_1 + \sum_{\lambda>1} 2m_\lambda = \dim(W). \tag{5}$$

Finally let $V \subset \mathscr{M}_\alpha$ be a finite dimensional σ-invariant subspace. Since $V^* \subset \mathscr{M}_{\alpha^c}$ it follows that $W = V + V^*$ is σ-invariant. If α is self-conjugate, then $W \subset \mathscr{M}_\alpha$, $W = W^*$ and therefore $\dim(V) \leq \dim(W) \leq M_\alpha^2$. In the case when $\alpha \neq \alpha^c$ in $\widehat{G}$ the sum $V + V^*$ is orthogonal and by (4) and (5) we get $2\dim(V) = \dim(W) \leq M_\alpha^2 + M_{\alpha^c}^2 = 2M_\alpha^2$. □

The next statement describes the modularity of ω.

Proposition 18. *There exists $\Theta : \mathscr{M}_0 \to \mathscr{M}_0$ linear multiplicative map such that $\omega(x\Theta(y)) = \omega(yx)$, $x \in \mathscr{M}$, $y \in \mathscr{M}_0$. Moreover Θ leaves invariant all the irreducible subspaces of $\mathscr{M}_0$ and is a scalar multiple of the modular operator F_α on such a subspace of $\mathscr{M}_\alpha$ for all $\alpha \in \widehat{G}$.*

Proof. Let $V_1,\dots,V_N$ be the mutually orthogonal irreducible σ-invariant subspaces of $\mathscr{M}_\alpha$ and let $e_1^{(k)},\dots,e_d^{(k)}$ be an orthonormal basis in V_k ($d=\dim(\alpha)$) such that $\sigma(e_i^{(k)})=\sum_{i=1}^d e_r^{(k)}\otimes u_{ri}^\alpha$, $1\le i\le d$, $1\le k\le N$. Then $\sum_{i=1}^d e_i^{(k)}e_i^{(l)\,*}\in\mathscr{M}^\sigma$, therefore there exist $\lambda_{kl}\in\mathbb{C}$, $1\le k,l\le N$ such that $\sum_{i=1}^d e_i^{(k)}e_i^{(l)\,*}=\lambda_{kl}1_{\mathscr{M}}$. The matrix $\Lambda=(\lambda_{kl})_{1\le k,l\le N}$ is positively definite, hence there exists a unitary $S=(s_{kl})_{1\le k,l\le N}\in\mathrm{Mat}_N(\mathbb{C})$ and $\lambda_1,\dots,\lambda_N>0$ such that $S\Lambda S^*=\mathrm{diag}(\lambda_1,\dots,\lambda_N)$. Replacing $e_i^{(k)}$ by $\sum_{p=1}^N s_{kp}e_i^{(p)}$ we get:

$$\sum_{i=1}^d e_i^{(k)}e_i^{(l)\,*}=\delta_{kl}\lambda_k 1_{\mathscr{M}},\qquad 1\le k,l\le N. \tag{6}$$

Fix now $1\le k,l\le N$ and consider $T_{kl}:V_k\to V_l$, $T_{kl}(x)=\sum_{j=1}^d\omega(xe_j^{(l)\,*})e_j^{(l)}$, $x\in V_k$. Combining the σ-invariance of ω, the orthogonality relations (5.25) in [Wor] and (6) we get for $1\le i\le d$:

$$\begin{aligned}
T_{kl}e_i^{(k)} &= \sum_{j=1}^d\omega(e_i^{(k)}e_j^{(\ell)\,*})e_j^{(l)}=\sum_{j,p,q=1}^d(\omega\otimes h)(e_p^{(k)}e_q^{(l)\,*}\otimes u_{pi}^\alpha u_{qj}^{\alpha\,*})e_j^{(l)}\\
&=\sum_{j,p,q=1}^d\omega(e_p^{(k)}e_q^{(\ell)\,*})\frac{\delta_{pq}}{M_\alpha}f_1(u_{ji}^\alpha)e_j^{(l)}=\sum_{j,p=1}^d\omega(e_p^{(k)}e_p^{(l)\,*})\frac{f_1(u_{ji}^\alpha)}{M_\alpha}e_j^{(l)}\\
&=\frac{\delta_{kl}\lambda_k}{M_\alpha}\sum_{j=1}^d f_1(u_{ji}^\alpha)e_j^{(k)}=\frac{\delta_{kl}\lambda_k}{M_\alpha}(id_{\mathscr{M}}\otimes f_1)(\sigma(e_i^{(k)})),
\end{aligned}$$

thus $T_{kl}=\frac{\delta_{kl}\lambda_k}{M_\alpha}(id_{\mathscr{M}}\otimes f_1)\sigma$. We check now that $T_k=T_{kk}\in\mathrm{Mor}(\sigma|_{V_k},\sigma^{cc}|_{V_k})$. The previous formula for T_k yields:

$$\sigma T_k=\frac{\lambda_k}{M_\alpha}\sigma((id_{\mathscr{M}}\otimes f_1)\sigma)=\frac{\lambda_k}{M_\alpha}(id_{\mathscr{M}}\otimes((f_1\otimes id_A)\Delta_A))\sigma$$

and

$$(T_k\otimes id_A)\sigma=\frac{\lambda_k}{M_\alpha}((id_{\mathscr{M}}\otimes f_1)\sigma\otimes id_A)\sigma=\frac{\lambda_k}{M_\alpha}(id_{\mathscr{M}}\otimes((f_1\otimes id_A)\Delta_A))\sigma.$$

Since $\kappa^2(a)=f_{-1}*a*f_1$, $a\in\mathscr{A}$ we also have:

$$\begin{aligned}
(id_{\mathscr{M}}\otimes\kappa^2)(id_{\mathscr{M}}\otimes(id_A\otimes f_1)\Delta_A)(y\otimes a)&=y\otimes\kappa^2(f_1*a)=y\otimes(a*f_1)\\
&=(id_{\mathscr{M}}\otimes(f_1\otimes id_A)\Delta_A)(y\otimes a),\qquad y\in\mathscr{M},a\in\mathscr{A},
\end{aligned}$$

hence $(T_k\otimes id_A)\sigma=(id_{\mathscr{M}}\otimes\kappa^2)\sigma T_k=\sigma^{cc}T_k$ on V_k.

But $\mathrm{Mor}(\alpha,\alpha^{cc})=\{\lambda F_\alpha\mid\lambda\in\mathbb{C}\}$, therefore $\omega(e_i^{(k)}e_j^{(l)\,*})=\delta_{kl}\lambda_k'(F_\alpha)_{ji}$. Comparing the previous relation with (6) we get:

$$\langle e_i^{(k)},e_j^{(l)}\rangle_{1,\omega}=\omega(e_i^{(k)}e_j^{(l)\,*})=\frac{\delta_{kl}\lambda_k}{M_\alpha}(F_\alpha)_{ji}=\frac{\delta_{kl}\lambda_k}{M_\alpha}\langle F_\alpha(e_i^{(k)}),e_j^{(l)}\rangle_{2,\omega},$$

thus $\omega(xy^*)=\delta_{kl}\omega(y^*\Theta(x))=\omega(y^*\Theta(x))$ for all $x\in V_k$, $y\in V_l$, where we let $\Theta|_{V_k}=\frac{\lambda_k}{M_\alpha}F_\alpha\in\mathscr{L}(V_k)$. The orthogonality of the spectral subspaces $\mathscr{M}_\alpha$ implies now that $\omega(xy)=\omega(y\Theta(x))$ for all $x\in\mathscr{M}_0$, $y\in\mathscr{M}$. Clearly Θ is linear by definition and follows multiplicative by the previous equality. ∎

2. The structure of the crossed product

Using the finite dimensionality of the spectral subspaces $\mathscr{M}_\alpha$ we prove that the reduced C^*-crossed product $\mathscr{N} = \mathscr{M} \times_\sigma \widehat{A}$ (as defined in [BS]) of a unital C^*-algebra $\mathscr{M}$ by an ergodic coaction of a compact matrix pseudogroup $G = (A, u)$ with faithful Haar measure h on A, turning $\mathscr{M}$ into an A-algebra, is isomorphic to a direct sum of algebras of compact operators, generalizing Proposition 2 in [L] and Corollary 2 in [Wa1, §1.4]. The main ingredient is the Takesaki-Takai duality type theorem of Baaj and Skandalis and the line of the proof follows Wassermann's one for the case when $A = C(G)$, the commutative C^*-algebra of continuous functions on a compact group G and $\mathscr{N} = \mathscr{M} \rtimes_\sigma G$.

Remark first that A is an A-algebra via the comultiplication $\Delta_A : A \to A \otimes A$, that Δ_A is ergodic and that h is the Δ_A-invariant state on A. Denote $\mathscr{H} = \mathscr{H}_h$ and consider as in Remark 15 the operator $V \in \mathscr{L}(\mathscr{H} \otimes \mathscr{H})$, $V(a_h \otimes b_h) = (\Delta_A(a)(1_A \otimes b))(1_h \otimes 1_h)$, $a, b \in A$. Then V is a biregular irreducible multiplicative unitary. Moreover, the partial isometry U (no relation with the unitary associated to the coaction σ in §1) in the polar decomposition $T = U|T|$ of the closure of the preclosed operator T_0 defined by $T_0(a_h) = \kappa(a)_h$, $a \in \mathscr{A}$ is a unitary on $\mathscr{H}$ with $U^2 = I_{\mathscr{H}}$. In fact $Ux_h = (f_1 * \kappa(x))_h$, $x \in \mathscr{A}$ and $(\mathscr{H}, V, U)$ is a Kac system (see [BS, §6]). Consider the reduced C^*-crossed product $\mathscr{N} = \mathscr{M} \times_\sigma \widehat{A}$, defined in [BS, §7] as the C^*-algebra generated by products of type $\sigma_L(x)(1_A \otimes \rho(\omega))$, $x \in \mathscr{M}$, $\omega \in \mathscr{L}(\mathscr{H})_*$. Then, there exists a dual coaction $\widehat{\sigma} = \sigma_{\mathscr{M} \times_\sigma \widehat{A}}$ on $\mathscr{N}$, which transforms this way into an $\widehat{A}$-algebra. Let A acting in the GNS representation of h on $\mathscr{H}$ and denote $\sigma_R(x) = (id_{\mathscr{M}} \otimes \mathrm{Ad}\, U)(\sigma(x))$, $x \in \mathscr{M}$, $\lambda(\omega) = (id_{\mathscr{L}(\mathscr{H}) \otimes \omega})(V)$, $\omega \in \mathscr{L}(\mathscr{H})_*$. Then, by Theorem 7.5 in [BS], $\mathscr{M} \times_\sigma \widehat{A}$ identifies with the norm closure of $\mathrm{span}\{\sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega)) \mid x \in \mathscr{M}, \omega \in \mathscr{L}(\mathscr{H})_*\}$, $\mathscr{M} \otimes \mathscr{K}(\mathscr{H})$ with the closure of $\mathrm{span}\{\sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega)a) \mid x \in \mathscr{M}, \omega \in \mathscr{L}(\mathscr{H})_*, a \in A\}$ and denoting $\sigma'(x \otimes k) = V_{23}\sigma(x)_{13}(1_{\mathscr{M}} \otimes k \otimes 1_A)V_{23}^*$, $x \in \mathscr{M}$, $k \in \mathscr{K}(\mathscr{H})$, the double crossed-product $((\mathscr{M} \times_\sigma \widehat{A}) \times_{\widehat{\sigma}} A, \widehat{\widehat{\sigma}})$ is isomorphic to $(\mathscr{M} \otimes \mathscr{K}(\mathscr{H}), \sigma')$ as A-algebras. Moreover, the last two equalities in the proof of that theorem show that via the previous identification $\mathscr{M} \times_\sigma \widehat{A} \subset (\mathscr{M} \otimes \mathscr{K}(\mathscr{H}))^{\sigma'}$.

The proof of the structure of the crossed-product will make use of the next Lemma:

Lemma. ([Wa1, §1.4]) *Let A be a (not necessarily unital) C^*-algebra with an approximate identity $p_1 \leq p_2 \leq \dots$ consisting of finite rank projections (i.e. $\dim(p_iAp_i) < \infty$). Then A is isomorphic to a direct sum of algebras of compact operators.*

Theorem 19. *Let $G = (A, u)$ be a compact matrixc pseudogroup and let $\mathscr{M}$ be a unital C^*-algebra which is an A-algebra via the ergodic coaction $\sigma : \mathscr{M} \to \mathscr{M} \otimes A$. Then $\mathscr{M} \times_\sigma \widehat{A} \simeq \bigoplus_i \mathscr{K}(\mathscr{H}_i)$.*

Proof. Replacing A by its reduced C^*-algebra, the crossed product $\mathscr{M} \times_\sigma \widehat{A}$ is still unchanged (we owe this remark to the referee), thus one may assume that the Haar measure is faithful on A. For each $\alpha \in \widehat{G}$ consider the projections $p(\alpha)$ from $\mathscr{H}$ onto $\mathscr{H}_\alpha$ defined in Remark 15. Since $p(\alpha) = (id_{\mathscr{L}(\mathscr{H})} \otimes \rho_\alpha)(V)$ and $\rho_\alpha \in \mathscr{L}(\mathscr{H})_*$ it follows that $p(\alpha) \in \widehat{A}$. But $\mathscr{M}$ is unital, therefore $1_{\mathscr{M}} \otimes p(\alpha) \in \mathscr{N} = \mathscr{M} \times_\sigma \widehat{A}$. Since $\sum_{\alpha \in \widehat{G}} p(\alpha) = 1_{\widehat{A}}$ in the strict topology, it follows that writing $\widehat{G} = \bigcup_n F_n$ with $\mathrm{card}(F_n) < \infty$ we get an approximate unit $p_n = p(F_n) = \sum_{\alpha \in F_n} p(\alpha)$, $n \geq 1$ of $\mathscr{N}$. Finally we check that each $p(F_n)$ has finite rank, or equivalently that $\dim(p(\alpha)\mathscr{N}p(\pi)) < \infty$

for all $\alpha, \pi \in \widehat{G}$. Since $1_{\mathscr{M}} \otimes p(\alpha) \in \mathscr{N}$ and $\mathscr{N} \subset (\mathscr{M} \otimes \mathscr{K}(\mathscr{H}))^{\sigma'}$, we obtain:

$$p(\alpha)\mathscr{N} p(\pi) \subset p(\alpha)(\mathscr{M} \otimes \mathscr{K}(\mathscr{H}))^{\sigma'} p(\pi) \subset (\mathscr{M} \otimes p(\alpha)\mathscr{K}(\mathscr{H})p(\pi))^{\sigma'} = (\mathscr{M} \otimes \mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha))^{\sigma'}.$$

If $\alpha_0 \in \widehat{G}$ is fixed, then $\sigma'|_{\mathscr{M}_{\alpha_0}^c \otimes \mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha)}$ is a corepresentation of A that coincides with the tensor product corepresentation $\sigma|_{\mathscr{M}_0^c} \odot \sigma_0|_{\mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha)}$ and therefore

$$\begin{aligned}(id_{\mathscr{M} \otimes \mathscr{L}(\mathscr{H})} \otimes h)(\sigma'(\mathscr{M}_0^c \otimes \mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha))) &= \operatorname{Ran}(\sigma|_{\mathscr{M}_0^c} \odot \sigma_0|_{\mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha)})_h \\ &= \operatorname{Mor}(\sigma|_{\mathscr{M}_0^c}, \sigma_0|_{\mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\sigma)})^T.\end{aligned}$$

Since there are only finitely many $\alpha_0 \in \widehat{G}$ that appear in $\sigma_0|_{\mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha)}$, the space $(\mathscr{M} \otimes \mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha))^{\sigma'} = (id_{\mathscr{M}} \otimes h)(\sigma'(\mathscr{M} \otimes \mathscr{L}(\mathscr{H}_\pi, \mathscr{H}_\alpha)))$ follows finite dimensional. □

Remark 20. *Although we didn't use it in proving the last theorem, it is easy to check that in fact* $\mathscr{M} \times_\sigma \widehat{A} = (\mathscr{M} \otimes \mathscr{K}(\mathscr{H}))^{\sigma'}$ *via the realization of the crossed-product in* $\mathscr{M} \otimes \mathscr{K}(\mathscr{H})$.

Proof. This is the case since the canonical conditional expectation $E = (id_{\mathscr{M} \otimes \mathscr{K}(\mathscr{H})} \otimes h)\sigma'$ carries total sets in $\mathscr{M} \otimes \mathscr{K}(\mathscr{H})$ onto total sets in $(\mathscr{M} \otimes \mathscr{K}(\mathscr{H}))^{\sigma'}$. In particular the set

$$\mathcal{S} = \{E(\sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega)a)) \mid x \in \mathscr{M}, \omega \in \mathscr{L}(\mathscr{H})_*, a \in A\}$$

is total in $(\mathscr{M} \otimes \mathscr{K}(\mathscr{H}))^{\sigma'}$. But the formula for σ' and $(id_A \otimes h)\Delta_A(a) = h(a)1_A$, $a \in A$, yield for any $x \in \mathscr{M}$, $\omega \in \mathscr{L}(\mathscr{H})_*$, $a \in A$:

$$\begin{aligned}E(\sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega)a)) &= (id_{\mathscr{M} \otimes \mathscr{K}(\mathscr{H})} \otimes h)((\sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega)) \otimes 1_A)(1_{\mathscr{M}} \otimes \Delta_A(a))) \\ &= \sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega))(1_{\mathscr{M}} \otimes (id_{\mathscr{L}(\mathscr{H})} \otimes h)\Delta_A(a)) \\ &= h(a)\sigma_R(x)(1_{\mathscr{M}} \otimes \lambda(\omega)),\end{aligned}$$

therefore $\mathcal{S} \subset \mathscr{M} \times_\sigma \widehat{A}$. □

Let B and C be unital C^*-algebras, $G = (A, u)$ be a compact matrix pseudogroup with A nuclear and $\sigma : C \to C \otimes A$ be a unital *-morphism. By the associativity of the maximal tensor product and the nuclearity of A, the diagram:

$$\begin{array}{ccc} B \otimes_{\max} C & \xrightarrow{\widetilde{\sigma} = id_B \otimes_{\max} \sigma} & B \otimes_{\max} (C \otimes A) = (B \otimes_{\max} C) \otimes A \\ \Big\downarrow \widetilde{\sigma} & & \Big\downarrow id_{B \otimes_{\max} C} \otimes \Delta_A = id_B \otimes_{\max} (id_C \otimes \Delta_A) \\ B \otimes_{\max} (C \otimes A) = (B \otimes_{\max} C) \otimes A & \xrightarrow{id_B \otimes_{\max} (\sigma \otimes id_A)} & (B \otimes_{\max} C) \otimes A \otimes A = B \otimes_{\max} (C \otimes A \otimes A) \end{array}$$

is commutative, although $\widetilde{\sigma}$ may not be one-to-one in general (note that $\widetilde{\sigma} \otimes_{\max} id_A = id_B \otimes_{\max} (\sigma \otimes id_A)$). But Lemma 4.1 i) holds true in such a case, thus $\widetilde{E} = (id_{B \otimes_{\max} C} \otimes h)\widetilde{\sigma}$ is a conditional expectation from $B \otimes_{\max} C$ onto $(B \otimes_{\max} C)^{\widetilde{\sigma}}$ and it turns out that $\widetilde{E} = id_B \otimes_{\max} E$, where $E = (id_C \otimes h)\sigma$ is conditional expectation from C onto C^σ. In particular, this shows

Remark 21. $(B \otimes_{\max} C)^{\widetilde{\sigma}} = B \otimes_{\max} C^\sigma$.

The next statement, whose proof follows the line of [Wa2, Lemma 22], shows that if A is nuclear, then $\mathscr{M}$ itself follows nuclear.

Proposition 22. *Let $(\mathscr{M},\sigma)$ be an A-algebra such that $\mathscr{M}\times_\sigma \widehat{A}$ and A are nuclear C^*-algebras. Then $\mathscr{M}$ is nuclear.*

Proof. We prove that given any unital C^*-algebra B, the natural *-morphism θ that maps $B\otimes_{\max}\mathscr{M}$ onto $B\otimes\mathscr{M}$ is one-to-one.

Consider the coaction $\sigma' : \mathscr{M}\otimes\mathscr{K}(\mathscr{H})\to\mathscr{M}\otimes\mathscr{K}(\mathscr{H})\otimes A$ and define as in the previous remark $\widetilde{\sigma'} : B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))\to B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))\otimes A$.

The map $\Theta=\theta\otimes id_{\mathscr{K}(\mathscr{H})} : B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))\to B\otimes\mathscr{M}\otimes\mathscr{K}(\mathscr{H})$ is A-equivariant, i.e. $(id_B\otimes\sigma')\Theta=\Theta\widetilde{\sigma'}$ and using the canonical conditional expectations onto fixed point algebras

$$E^{\widetilde{\sigma'}} : B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))\to(B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H})))^{\widetilde{\sigma'}}=B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))^{\widetilde{\sigma'}}$$

and

$$E^{id_B\otimes\sigma'} : B\otimes\mathscr{M}\otimes\mathscr{K}(\mathscr{H})\to(B\otimes\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))^{id_B\otimes\sigma'}=B\otimes(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))^{\sigma'},$$

we get $\Theta E^{\widetilde{\sigma'}}=E^{id_B\otimes\sigma'}\Theta$. Consequently

$$(\operatorname{Ker}\theta\otimes\mathscr{K}(\mathscr{H}))^{\widetilde{\sigma'}}=\operatorname{Ker}\Big(B\otimes_{\max}(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))^{\sigma'}\xrightarrow{\Theta}B\otimes(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))^{\sigma'}\Big).$$

Since $(\mathscr{M}\otimes\mathscr{K}(\mathscr{H}))^{\sigma'}\simeq\mathscr{M}\times_\sigma\widehat{A}$ is nuclear, it follows that $(\operatorname{Ker}\theta\otimes\mathscr{K}(\mathscr{H}))^{\widetilde{\sigma'}}=0$ and therefore $\operatorname{Ker}\theta=0$. □

Corollary 23. *If $G=(A,u)$ is a compact matrix pseudogroup, the C^*-algebra A is nuclear and $\mathscr{M}$ is a unital C^*-algebra which is an A-algebra via an ergodic coaction, then $\mathscr{M}$ is nuclear.*

References

[BS] S. Baaj, G. Skandalis, *Unitaires multiplicatifs at dualité pour les produits croisés de C^*–algèbres*, Ann. Sci. Ec. Norm. Sup. **26** (1993), 425–488.

[HLS] R. Høegh-Krohn, M. Landstad, E. Størmer, *Compact ergodic groups of automorphisms*, Ann. Math. **114** (1981), 75–86.

[L] M. Landstad, *Algebras of spherical functions associated with covariant systems*, Math. Scandinavica **47** (1980), 137–149.

[N] G. Nagy, *On the Haar measure on the twisted* SU(N) *group*, Comm. Math. Phys. **153** (1993), 217–228.

[T] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, 1979.

[Wa1] A. Wassermann, *Ergodic actions of compact groups on operator algebras. I. General theory*, Ann. Math. **130** (1989), 273–319.

[Wa2] A. Wassermann, *II. Classification of full multiplicity actions*, Can. J. Math. **40** (1988), 1482–1527.

[Wa3] A. Wassermann, *III. Classification for* SU(2), Invent. Math. **93** (1988), 309–355.

[Wa4] A. Wassermann, *Coactions and Yang-Baxter equations for ergodic actions and subfactors*, L.M.S. Lecture Notes **136** (1988), 202–236.

[Wor] S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90024-1555

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 70700, Bucharest, Romania

Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1, Canada