AN AF ALGEBRA ASSOCIATED WITH THE FAREY TESSELATION

FLORIN P. BOCA

Abstract. To the Farey tesselation of the upper half-plane we associate an AF algebra \( A \) encoding the ‘cutting sequences’ that define vertical geodesics. The Effros-Shen AF algebras arise as quotients of \( A \). Using the path model for AF algebras we construct, for each \( \tau \in (0, \frac{1}{2}] \), projections \( (E_n)_n \) in \( A \) such that \( E_n E_{n+1} \leq \tau E_n \).

Contents

Introduction 1
1. The Pascal triangle with memory as a Bratelli diagram 3
2. The primitive ideal space of the AF algebra \( A \) 5
3. The Jacobson topology on \( \text{Prim} A \) 11
4. A description of the dimension group 14
5. Traces on \( A \) 16
6. Generators, relations, and braiding 19
Acknowledgments 23
References 23

Introduction

The semigroup \( S \) generated by the matrices \( A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) is isomorphic to \( \mathbb{F}_2^+ \), the free semigroup on two generators. This fact, intimately connected to the continued fraction algorithm, can be visualized by means of the Farey tesselation \( gG : g \in \mathbb{S} \) of \( \mathbb{H} \) depicted in Figure 1, where \( \mathbb{G} = \{ 0 \leq \Re z \leq 1, |z - \frac{1}{2}| \geq \frac{1}{2} \} \) (cf., e.g., [25]).

The strip \( 0 \leq \Re z \leq 1 \) is tesselated precisely by the images of \( \mathbb{G} \) under matrices from the set

\[
\mathbb{S}_* = \{ I \} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : 0 \leq a \leq c, 0 \leq b \leq d \right\}.
\]

By suspending the cusps in this tesselation (which correspond to rational numbers in \( [0, 1] \)) with appropriate (infinite) multiplicities, one gets the diagram \( \mathcal{G} \) from Figure 2 (cf. [19]). This diagram reflects both the elementary median construction, that produces from a pair \( \left( \frac{p}{q}, \frac{p'}{q'} \right) \) of rational numbers with \( p'q - pq' = 1 \) the new pairs \( \left( \frac{p}{q}, \frac{p+q'}{q+q} \right) \) and \( \left( \frac{p+p'}{q+q'}, \frac{p'}{q'} \right) \) with the same property, and the “geometry” of the continued fraction algorithm. As in the case of the Pascal triangle, in \( \mathcal{G} \) one writes the sum of the denominators of two neighbors from the same floor into the next floor of the diagram.

Date: August 29, 2006.
2000 Mathematics Subject Classification. Primary: 46L05; Secondary: 11A55, 11B57, 46L55, 37E05, 82B20.
One keeps, however, a copy of each denominator at the next floor. For this reason, such a diagram was called the Pascal triangle with memory [18]. There is a remarkable one-to-one correspondence between the integer solutions of the equation $ad - bc = 1$ with $0 \leq a \leq c$, $0 \leq b \leq d$, and the rational labels of two neighbors at the same floor in $\mathcal{G}$, acquired by the mediant construction and by keeping each label at the next floor in the diagram.

The thrust of this paper is the remark that, by regarding $\mathcal{G}$ as a Bratteli diagram, one gets an AF algebra $\mathfrak{A} = \lim\limits_{\rightarrow} \mathfrak{A}_n$ with interesting properties. This algebra is closely related with the Effros–Shen AF algebras [10, 21] which we show to arise as primitive quotients of $\mathfrak{A}$. The primitive ideal space $\text{Prim} \mathfrak{A}$ is identified with the disjoint union of the irrational numbers in $[0, 1]$ and three copies of the rational ones, except for the endpoints 0 and 1 which are only represented by two copies.

In [3] it was shown that any separable abelian $C^*$-algebra $\mathfrak{Z}$ is the center $Z(\mathfrak{A})$ of an AF algebra $\mathfrak{A}$. The AF algebra $\mathfrak{A}$ can actually be retrieved from that abstract construction by embedding $\mathfrak{Z} = C[0, 1]$ into the norm closure in $L^\infty[0, 1]$ of the linear space of the characteristic functions of open sets $(\frac{k}{2^n}, \frac{k+1}{2^n})$ and of singleton sets $\{\frac{\ell}{2^n}\}$, $n \geq 0$, $0 \leq k < 2^n$, $0 \leq \ell \leq 2^n$. In particular this shows that $Z(\mathfrak{A}) = C[0, 1]$.

The connecting maps $K_0(\mathfrak{A}_n) \hookrightarrow K_0(\mathfrak{A}_{n+1})$ correspond to the polynomial relations $p_{n+1}(t) = (1 + t + t^2)p_n(t^2)$. These polynomials are closely related to the Stern–Brocot sequence (cf. [6]). The origins of this remarkable sequence, which has attracted considerable interest in time, can be traced back to Eisenstein (see [27], [5], or the contemporary reference [26] for a thorough bibliography on this subject). In our framework the Stern–Brocot sequence $q(n, k)$, $n \geq 0$, $0 \leq k < 2^n$, simply appears as $\mathfrak{A}_n = \bigoplus_{k=0}^{2^n-1} M_{q(n, k)} \otimes \mathbb{C}$.

The Bratteli diagram $\mathcal{G}$ has some apparent symmetries. In the last section we employ the AF path model to express them, constructing sequences of projections in $\mathfrak{A}$ that satisfy certain braiding relations reminiscent of the Temperley-Lieb-Jones relations. In particular, for every $\tau \in (0, \frac{1}{4}]$, we construct projections $E_n$ in $\mathfrak{A}$, $n \geq 0$, such that $E_n E_{n \pm 1} E_n \leq \tau E_n$ and $[E_n, E_m] = 0$ if $|n - m| \geq 2$. This suggests a possible connection with a class of statistical mechanics models with partition functions closely related to Riemann’s zeta function, called Farey spin chains, that have been studied in recent
showing in particular that \( p \) and \( q \) years by Knauf, Kleban, and their collaborators (see, e.g. [17, 18, 19, 16, 22] and references therein).

1. The Pascal triangle with memory as a Bratelli diagram

The Pascal triangle with memory is a graph \( G = (\mathcal{V}, \mathcal{E}) \) defined as follows:

- The vertex set \( \mathcal{V} \) is the disjoint union \( \bigcup_{n \geq 0} \mathcal{V}_n \) of the sets \( \mathcal{V}_n = \{(n, k) : 0 \leq k \leq 2^n \} \) of vertices at floor \( n \);
- The set of edges is defined as \( \mathcal{E} = \bigcup_{n \geq 0} \mathcal{E}_n \), where \( \mathcal{E}_n \) is the set of edges connecting vertices at floor \( n \) with those at floor \( n + 1 \) under the rule that \((n, k)\) is connected with \((n + 1, \ell)\) precisely when \(|2k - \ell| \leq 1\). There are no edges connecting vertices from \( \mathcal{V}_i \) and \( \mathcal{V}_j \) when \(|i - j| \geq 2\).

To each vertex \((n, k)\) we attach the label \( r(n, k) = \frac{p(n, k)}{q(n, k)} \), with non-negative integers \( p(n, k), q(n, k) \) defined recursively for \( n \geq 0 \) by

\[
\begin{aligned}
q(n, 0) &= q(n, 2^n) = 1, \quad p(n, 0) = 0, \quad p(n, 2^n) = 1; \\
q(n + 1, 2k) &= q(n, k), \quad p(n + 1, 2k) = p(n, k), \quad 0 \leq k \leq 2^n; \\
q(n + 1, 2k + 1) &= q(n, k) + q(n, k + 1), \\
p(n + 1, 2k + 1) &= p(n, k) + p(n, k + 1), \quad 0 \leq k < 2^n.
\end{aligned}
\]

Note that \( r(n, 0) = 0 < r(n, 1) = \frac{1}{n+1} < \cdots < r(n, 2^n) = 1 \) gives a partition of \([0, 1]\), and

\[ p(n, k + 1)q(n, k) - p(n, k)q(n, k + 1) = 1, \quad n \geq 0, \quad 0 \leq k < 2^n, \]

showing in particular that \( p(n, k) \) and \( q(n, k) \) are relatively prime.

\[ \begin{array}{c}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
34 & 34 & 34 & 34 & 34 & 34 & 34 & 34 & 34 \\
55 & 55 & 55 & 55 & 55 & 55 & 55 & 55 & 55 \\
89 & 89 & 89 & 89 & 89 & 89 & 89 & 89 & 89 \\
144 & 144 & 144 & 144 & 144 & 144 & 144 & 144 & 144 \\
\end{array} \]

Conversely, for every pair \( \frac{p}{q} < \frac{p'}{q'} \) of rational numbers with \( p'q - pq' = 1, \ 0 \leq p \leq q \) and \( 0 \leq p' \leq q' \), there exists a unique pair of integers \((n, k)\) with \( n \geq 0, \ 0 \leq k < 2^n \), such that \( r(n, k) = \frac{p}{q} \) and \( r(n, k + 1) = \frac{p'}{q'} \). This correspondence establishes a bijection between the vertices from \( \mathcal{V} \setminus \{(n, 2^n) : n \geq 0\} \) and the set

\[ \Gamma^+ := \left\{ \left( \frac{p'}{q'}, \frac{p}{q} \right) \in SL_2(\mathbb{Z}) : 0 \leq p \leq q, \ 0 \leq p' \leq q' \right\} \subset \Gamma := SL_2(\mathbb{Z}). \]
Remark 1. The mapping \( r(n,k) \mapsto \frac{k}{2^n}, 0 \leq k \leq 2^n, n \geq 0 \), extends by continuity to Minkowski’s map \(? : [0,1] \to [0,1] \) defined on (reduced) continued fractions as
\[
?(a_1,a_2,\ldots) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{2^{(a_1+\cdots+a_k)-1}}.
\]
The map \(? \) is strictly increasing and singular, and establishes remarkable one-to-one correspondences between rational and dyadic numbers, and respectively between quadratic irrationals and rational numbers in \([0,1]\) (see \([20, 7, 24]\)).

In this paper we shall consider the AF algebra \( \mathfrak{A} \) associated with the Bratteli diagram \( D(\mathfrak{A}) = \mathcal{G} \) from Figure 2. For the connection between Bratteli diagrams, AF algebras, and their ideals, we refer to the classical reference \([1]\). We write \((n,k)\) for quadratic irrationals and rational numbers in \([0,1]\).

The mapping \( \Phi : (n,k) \mapsto \mathcal{G} \) is strictly increasing and singular, and establishes remarkable one-to-one correspondences between rational and dyadic numbers, and respectively between quadratic irrationals and rational numbers in \([0,1]\) (see \([20, 7, 24]\)).

Remark 2. Consider the set \( \mathcal{V}_s \) of vertices of \( \mathcal{G} \) of form \((n,k)\) with \(0 \leq k \leq 2^n\) and \(k \) odd, and the map \( \Phi : \mathcal{V}_s \to \mathbb{N}, \Phi(n,k) = q(n,k) \). The inverse image \( \Phi^{-1}(q) \) of \( q \) contains exactly \( \varphi(q) \) elements, where \( \varphi \) denotes Euler’s totient function; in particular \( q \) is prime if and only if \( \#\Phi^{-1}(q) = q − 1 \). This remark shows, cf. \([17]\), that the partition function associated with the corresponding Farey spin chain is \( \sum_{s=1}^{\infty} \varphi(n)n^{-s} \), which is equal to \( \zeta(s−1)/\zeta(s) \) when \( \Re s > 2 \).

Remark 3. (i) The integers \( q(n,k) \) satisfy the equality
\[
\sum_{0 \leq k \leq 2^n} q(n,k) = 3^n + 1.
\]

(ii) Consider the Bratteli diagram obtained by deleting in \( \mathcal{G} \) all vertices labelled by \( 0 \) and denote the corresponding AF algebra by \( \mathfrak{B} = \varprojlim \mathfrak{B}_n \). It is clear that \( \mathfrak{B} \) is an ideal in \( \mathfrak{A} \) and \( \mathfrak{A}/\mathfrak{B} \cong \mathbb{C} \). Moreover,
\[
\mathfrak{B}_n = \bigoplus_{1 \leq k \leq 2^n} \mathbb{M}_{p(n,k)},
\]

thus the ranks of the central summands of the building blocks of \( \mathfrak{B} \) give the complete list of numerators \( p(n,k) \).

We also have
\[
\sum_{0 \leq k \leq 2^n} p(n,k) = \frac{3^n + 1}{2}.
\]
2. The primitive ideal space of the AF algebra \( A \)

We denote
\[
\mathbb{I} = \{ \theta \in (0, 1) : \theta \notin \mathbb{Q} \}, \quad \mathbb{Q}_{(0,1)} = \mathbb{Q} \cap (0, 1).
\]

The \( C^* \)-algebra \( A \) is not simple and has a rich (and potentially interesting) structure of ideals. We first relate \( A \) with the AF algebra \( F_\theta \) associated by Effros and Shen [10] to the continued fraction decomposition \( \theta = [a_1, a_2, \ldots] \) of \( \theta \in \mathbb{I} \). The Bratteli diagram \( D(\mathcal{F}_\theta) \) of the simple \( C^* \)-algebra \( F_\theta \) is given in Figure 3.

\[
\begin{array}{c}
\bullet \quad a_1 \quad \bullet \quad a_2 \quad \bullet \quad a_3 \quad \bullet \quad a_4 \quad \cdots \\
\end{array}
\]

\textbf{Figure 3.} The Bratteli diagram \( D(\mathcal{F}_\theta) \).

The \( C^* \)-algebra of unitized compact operators \( \widetilde{K} = \mathbb{C}I + K \) is an AF algebra and we have a short exact sequence \( 0 \to K \to \widetilde{K} \to \mathbb{C} \to 0 \), made explicit by the Bratteli diagram in Figure 4, where the shaded subdiagram corresponds to the ideal \( K \). Replacing \( \mathbb{C} \oplus \mathbb{C} \) by \( M_q \oplus M_q' \) one gets an AF algebra \( A_{q,q'} \) which is an extension of \( K \) by \( M_q \).

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \\
\end{array}
\]

\textbf{Figure 4.} The Bratteli diagram of the \( C^* \)-algebra of unitized compact operators.

We first show that Effros-Shen algebras arise naturally as quotients of our AF algebra \( A \) and that the corresponding ideals belong to the primitive ideal space \( \text{Prim } A \). To state a precise result we need to consider the \textit{Farey map} \( F : [0, 1] \to [0, 1] \) defined [14] by
\[
F(x) = \begin{cases} 
\frac{x}{1-x} & \text{if } x \in [0, \frac{1}{2}] , \\
\frac{1-x}{x} & \text{if } x \in (\frac{1}{2}, 1].
\end{cases}
\] (2.1)

This map acts on infinite (reduced) continued fractions as
\[
F([a_1, a_2, a_3, \ldots]) = [a_1 - 1, a_2, a_3, \ldots].
\]

For each \( y \in [0, 1] \) the equation \( F(x) = y \) has exactly two solutions \( x \in [0, 1] \) given by
\[
x = F_1(y) = \frac{y}{1+y} \quad \text{and} \quad x = F_2(y) = \frac{1}{1+y} = 1 - F_1(y).
\] (2.2)

One has \( F_1([a_1, a_2, \ldots]) = [a_1 + 1, a_2, \ldots] \) and \( F_2([a_1, a_2, \ldots]) = [1, a_1, a_2, \ldots] \). Rational numbers are generated by the backwards orbit of \( F \) as follows:
\[
\{ F^{-n}(\{0\}) : n = 0, 1, 2, \ldots \} = \mathbb{Q} \cap [0, 1].
\]
More precisely, for each \( n \in \mathbb{N} \) one has

\[
F^{-n}(\{0\}) = \{ r(n-1,k) : 0 \leq k \leq 2^{n-1} \} = \{ F_{i_1}^{\alpha_1} \ldots F_{i_k}^{\alpha_k}(0) : i_j \in \{1,2\}, i_1 \neq \ldots \neq i_k, \alpha_1 + \ldots + \alpha_k = n \} = \{ [a_1, \ldots, a_r] : a_1 + \ldots + a_r \leq n \}.
\]

In the next statement, given \( 0 < p < q \) relatively prime integers, \( \overline{p} \) will denote the multiplicative inverse of \( p \) modulo \( q \), i.e. the unique integer \( \overline{p} \in \{1, \ldots, q-1\} \) with \( p\overline{p} = 1 \mod q \).

**Proposition 4.** (i) For each \( \theta \in \mathbb{I} \), there is \( I_0 \in \text{Prim} \mathfrak{A} \) such that \( \mathfrak{A}/I_0 \cong \mathfrak{A}(\theta) \).

(ii) Given \( p/q \in \mathbb{Q}(0,1) \) in lowest terms, there are \( I_0^+ , I^- \in \text{Prim} \mathfrak{A} \) such that \( \mathfrak{A}/I_0^+ \cong \mathbb{M}_q, \mathfrak{A}/I^- \cong \mathfrak{A}(q,p) \), and \( \mathfrak{A}/I_0 \cong \mathfrak{A}(q,q-p) \).

(iii) There are \( I_0^+, I_0^-, I_1^+ , I_1^- \in \text{Prim} \mathfrak{A} \) such that \( \mathfrak{A}/I_0^+ \cong \mathfrak{A}/I_1^+ \cong \mathbb{C} \) and \( \mathfrak{A}/I_0^- \cong \mathfrak{A}/I_1^- \cong \mathfrak{A}/I_0^+ \cong \mathfrak{A}/I_1^- \cong \mathfrak{K} \).

**Proof.** (i) Let \( \theta \in \mathbb{I} \) with continued fraction \([a_1, a_2, \ldots]\) and \( r_\ell = r_\ell(\theta) = p_\ell/q_\ell = [a_1, \ldots, a_\ell] \) its \( \ell^{\text{th}} \) convergent, where \( p_\ell = p_\ell(\theta) \) and \( q_\ell = q_\ell(\theta) \) can be recursively defined by

\[
\begin{cases}
p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = 0, \quad q_0 = 1; \\
\begin{pmatrix} p_\ell & q_\ell \\ p_{\ell-1} & q_{\ell-1} \end{pmatrix} = \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{\ell-1} & q_{\ell-1} \\ p_{\ell-2} & q_{\ell-2} \end{pmatrix}, \quad \ell \geq 1.
\end{cases}
\]

The relation \( p_\ell q_{\ell-1} - p_{\ell-1} q_\ell = (-1)^{\ell-1} \) shows in particular that \( \gcd(p_\ell, q_\ell) = 1 \).

![Figure 5. The diagrams \( L_a \) and \( R_a \).](image)

For each \( a \in \mathbb{N} = \{1, 2, \ldots\} \) consider the diagrams \( L_a \) and \( R_a \) from Figure 5. Also set \( L_0 = R_0 = \emptyset \). Clearly \( L_{a+b} \) coincides with the concatenation \( L_a \circ L_b \) of \( L_a \) followed by \( L_b \), and we also have \( R_{a+b} = R_a \circ R_b \). Using the obvious identifications between \( L_a \circ R_b, R_a \circ L_b \), and \( C_a \circ C_b \) (see Figure 6), we see that

\[
L_{a_1} \circ R_{a_2} \circ L_{a_3} \circ R_{a_4} \circ \ldots = R_{a_1} \circ L_{a_2} \circ R_{a_3} \circ L_{a_4} \circ \ldots = C_{a_2} \circ C_{a_3} \circ C_{a_4} \circ \ldots = D(\mathfrak{F}[a_1,a_2,a_3,\ldots]).
\]

For each \( a \in \mathbb{N} \) consider the diagrams \( L(a) := L_{a-1} \circ R_1 \) and respectively \( R(a) := R_{a-1} \circ L_1 \) (see Figure 7). For each irrational number \( \theta = [a_1, a_2, \ldots] \) we extract from \( \mathcal{G} \) the subdiagram \( \mathcal{G}_\theta \) that contains the vertices \((0,0)\) and \((0,1)\), and is defined by the Bratteli diagram

\[
L(a_1) \circ R(a_2) \circ L(a_3) \circ R(a_4) \circ \ldots = L_{a_1-1} \circ R_{a_2} \circ L_{a_3} \circ R_{a_4} \circ \ldots = D(\mathfrak{F}[a_1-1,a_2,a_3,\ldots]) = D(\mathfrak{F}(\theta)).
\]
The complement \( G \setminus G_{\theta} \) is a directed and hereditary Bratteli diagram as in [1, Lemma 3.2] (see also Figure 8). Thus there is an ideal \( I_{\theta} \) in \( \mathfrak{A} \) such that \( D(I_{\theta}) = G \setminus G_{\theta} \), \( D(\mathfrak{A}/I_{\theta}) = G_{\theta} \), and \( \mathfrak{A}/I_{\theta} \cong \mathfrak{F}(\theta) \). Moreover \( I_{\theta} \) is a primitive ideal cf. [1, Theorem 3.8].

If \( j_{n} = j_{n}(\theta) \) is the unique index for which \( r(n,j_{n}) < \theta < r(n,j_{n} + 1) \) (see Figure 8), then

\[
I_{\theta} \cap \mathfrak{A}_{n} = \bigoplus_{0 \leq k \leq 2^{n}} \mathbb{M}_{q(n,k)}. 
\]

The vertices of \( D(\mathfrak{A}/I_{\theta}) \) are explicitly related to the continued fraction decomposition of \( \theta \). For each \( r \in \mathbb{Q}_{(0,1)} \), denote \( h(t) = \min\{n : \exists k, \ r(n,k) = r\} \). Let \( \frac{p_{n}}{q_{n}} \) be the continued fraction approximations of \( \theta \), and \( h_{n} = \min\{h(\frac{p_{n}}{q_{n}})\} \). With this notation, the labels of the two vertices at floor \( m \) in \( G_{\theta} \) are \( \frac{p_{m}}{q_{m}} \) and \( \frac{p_{m-1} + (m - h_{m})p_{n}}{q_{m-1} + (m - h_{m})q_{n}} \) whenever \( h_{n} \leq m < h_{n+1} \).

(ii) For each \( \theta = \frac{p}{q} \in \mathbb{Q}_{(0,1)} \) in lowest terms, consider the Bratteli subdiagram \( G_{\theta} \) of \( G \) defined by all vertices \( (n,j) \) with \( r(n,j) = \theta \) and \( (m,i) \) with \( (m,i) \downarrow (n,j) \) (see Figure 9). The AF algebra associated to \( G_{\theta} \) is clearly isomorphic to \( \mathbb{M}_{q} \). Again, the complement \( G \setminus G_{\theta} \) is seen to be a directed and hereditary Bratteli diagram. Therefore there is a primitive ideal \( I_{\theta} \) in \( \mathfrak{A} \) such that \( D(I_{\theta}) = G \setminus G_{\theta} \) and \( \mathfrak{A}/I_{\theta} \cong \mathbb{M}_{q} \).

Let \( n_{0} - 1 = n_{0}(\theta) - 1 \) be the largest \( n \in \mathbb{N} \) for which there exists \( j = j_{n}(\theta) \) such that \( r(n,j) < \theta < r(n,j + 1) \). For \( n < n_{0} \) define \( j_{n} \) as above. By the choice of \( n_{0} \) and the

Figure 6. The identification between \( L_{a} \circ R_{b} \), \( R_{a} \circ L_{b} \), and \( C_{a} \circ C_{b} \).

Figure 7. The diagrams \( L(a) \) and \( R(a) \).
properties of the Pascal triangle with repetition, for every \(n \geq n_0\) there is \(j_n = j_n(\theta)\) with \(r(n, j_n) = \theta\). The ideal \(I_{\theta}\) is generated by the direct summands \(M_{q(n_0,j_{n_0}-1)}\), \(M_{q(n_0,j_{n_0}+1)}\) and \(M_{q(n,c_n)}\), \(n < n_0\), that is

\[
I_{\theta} \cap \mathfrak{A}_n = \begin{cases} 
\bigoplus_{0 \leq k \leq 2^n} M_{q(n,k)} & \text{if } n < n_0, \\
\bigoplus_{0 \leq k \leq 2^n} M_{q(n,k)} & \text{if } n \geq n_0.
\end{cases}
\]

The ideals \(I^+_{\theta}\) defined by (see also Figures 10 and 11)

\[
I^+_{\theta} \cap \mathfrak{A}_n = \bigoplus_{0 \leq k \leq 2^n} M_{q(n,k)},
\]

and respectively by

\[
I^-_{\theta} \cap \mathfrak{A}_n = \begin{cases} 
\bigoplus_{0 \leq k \leq 2^n} M_{q(n,k)} & \text{if } n < n_0, \\
\bigoplus_{0 \leq k \leq 2^n} M_{q(n,k)} & \text{if } n \geq n_0,
\end{cases}
\]

are primitive and we clearly have \(\mathfrak{A}/I^-_{\theta} \cong \mathfrak{A}(q,p)\) and \(\mathfrak{A}/I^+_{\theta} \cong \mathfrak{A}(q,q-p)\).
Remark 5. A joint (and important) feature of all cases above is that
\[(n, j) \notin D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta \implies r(n, j - 1) < \theta < r(n, j + 1).\]

Remark 6. In $\text{SL}_2(\mathbb{Z})$ consider the matrices
\[A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.\]
The identification between $L_aR_b$ and $C_aC_b$ reflects the matrix equality

$$B^aA^b = M(a)M(b),$$

whereas the identification between $R_aR_b$ and $C_aC_b$ reflects the matrix equality

$$A^aB^b = JM(a)M(b)J.$$

A combinatorial analysis based on Bratteli’s correspondence between primitive ideals and subdiagrams of $\mathcal{G}$ shows that these are actually the only primitive ideals of $\mathfrak{A}$.

**Proposition 7.** \(\text{Prim } \mathfrak{A} = \{I_0 : \theta \in 1\} \cup \{I_0, I_0^\pm : \theta \in \mathbb{Q}_{(0,1)}\} \cup \{I_0, I_0^+, I_1, I_1^\pm\}\).

**Proof.** Let $I \in \text{Prim } \mathfrak{A}$. Consider the Bratteli diagrams $D = D(I)$ and $\tilde{D} = D(\mathfrak{A}/I) = \mathcal{G} \setminus D$. If there is $n_0$ such that $(n_0, k) \in D$ for all $0 \leq k \leq 2^{n_0}$, then $I = \mathfrak{A}$. So for each $n$ the set $L_n = \{k : (n, k) \in \tilde{D}\}$ is nonempty. Denote also $L_n^{\pm} = \{0, 1, \ldots, 2^n\} \setminus L_n$.

We first notice that $L_n$ should be a set of the form $\{a_n\}$ or $\{a_n, a_n + 1\}$. If not, there are $k, k' \in L_n$ such that $k' - k \geq 2$. Since $I$ is a primitive ideal, a vertex $(p, r)$ such that $(n, k) \not\leq (p, r)$ and $(n, k') \not\leq (p, r)$ in $\mathcal{G}$ should exist. Since $k' - k > 2$ this is not possible due to the definition of $\mathcal{G}$.

To finish the proof it suffices to show that

$$L_{n+1} = \begin{cases} \{2a_n\} & \text{if } L_n = \{a_n\}, \\ \{2a_n, 2a_n + 1\}, \{2a_n + 1, 2a_n + 2\}, & \text{if } L_n = \{a_n, a_n + 1\}, \\ \text{or } \{2a_n + 1\} & \end{cases} \quad (2.3)$$

that is, all links $(n, j) \not\leq (n + 1, j')$ in $\tilde{D}$ are exactly as indicated in Figure 12.

Indeed, if $L_n = \{a_n\}$, then $(n, a_n - 1), (n, a_n + 1)$ are vertices in the hereditary diagram $D$; thus we also have $(n + 1, 2a_n - 1), (n + 1, 2a_n + 1) \in D$. Because $D$ is directed, $(n + 1, 2a_n) \in D$ would imply $(n, a_n) \in D$, which contradicts $a_n \in L_n$. 

---

**Figure 11.** The diagrams $D(I_5^-)$ (darker) and $D(\mathfrak{A}/I_5^-)$ (lighter).
If \( L_n = \{a_n, a_n + 1\} \), then \( (n, a_n - 1), (n, a_n + 2) \in D \). Moreover because \( D \) is hereditary the vertices \( (n+1, 2a_n-1) \) and \( (n+1, 2a_n+3) \) also belong to \( D \). We now look at the consecutive vertices \( (n+1, 2a_n), (n+1, 2a_n+1), (n+1, 2a_n+2) \). From the first part they cannot all belong to \( \tilde{D} \). If \( (n+1, 2a_n+1) \in D \), and \( (n+1, 2a_n), (n+1, 2a_n+2) \in \tilde{D} \), then \( L_{n+1} \) has a gap, thus contradicting the first part. If \( (n+1, 2a_n), (n+1, 2a_n+1) \in D \) it follows, as a result of the fact that \( (n+1, 2a_n-1) \in D \) and that \( D \) is directed, that \( (n+1, 2a_n+1) \in \tilde{D} \). In a similar way one cannot have \( (n+1, 2a_n+1), (n+1, 2a_n+2) \in D \). It remains that only the following cases can occur (see also Figure 12):

(i) \( (n+1, 2a_n), (n+1, 2a_n+1) \in \tilde{D} \) and \( (n+1, 2a_n+2) \in D \), thus \( L_{n+1} = \{2a_n, 2a_n + 1\} \).

(ii) \( (n+1, 2a_n) \in D \) and \( (n+1, 2a_n+1), (n+1, 2a_n+2) \in \tilde{D} \), thus \( L_{n+1} = \{2a_n+1, 2a_n + 2\} \).

(iii) \( (n+1, 2a_n+1) \in \tilde{D} \) and \( (n+1, 2a_n), (n+1, 2a_n+2) \in D \), thus \( L_{n+1} = \{2a_n+1\} \), which concludes the proof of (2.3).

\[\begin{array}{c}
\text{Figure 12. The possible links between two consecutive floors in } D(\mathfrak{A}/I), I \in \text{Prim} \mathfrak{A}.
\end{array}\]

3. The Jacobson topology on Prim \( \mathfrak{A} \)

We first recall some basic things about the primitive ideal space of a \( C^* \)-algebra \( \mathcal{A} \) following [8] and [23]. For each set \( S \subseteq \text{Prim} \mathcal{A} \), consider the ideal \( k(S) := \bigcap_{J \in S} J \) in \( \mathcal{A} \), called the kernel of \( S \). For each ideal \( I \) consider its hull, \( h(I) := \{P \in \text{Prim} \mathcal{A} : I \subseteq P\} \). The closure of a set \( S \subseteq \text{Prim} \mathcal{A} \) is defined as

\[ S^\circ := \{P \in \text{Prim} \mathcal{A} : k(S) \subseteq P\}. \]

There is a unique topology on \( \text{Prim} \mathcal{A} \), called the Jacobson (or hull-kernel) topology such that its closed sets are exactly those with \( S = S^\circ \). The open sets in \( \text{Prim} \mathcal{A} \) are then precisely those of the form

\[ O_I := \{P \in \text{Prim} \mathcal{A} : I \not\subseteq P\} \]

for some ideal \( I \) in \( \mathcal{A} \). The Jacobson topology is always \( T_0 \), i.e. for any two distinct points in \( \text{Prim} \mathcal{A} \) one of them has a neighborhood which does not contain the other.

Moreover, the correspondence \( S \mapsto k(S) \) establishes a one-to-one correspondence between the closed subsets \( S \) of \( \text{Prim} \mathcal{A} \) and the lattice of ideals in \( \mathcal{A} \), with inverse given by \( I \mapsto h(I) \). For any ideal \( I \) in \( \mathcal{A} \), let \( p_I \) denote the quotient map \( \mathcal{A} \to \mathcal{A}/I \). The mapping \( P \mapsto P \cap I \) is a homeomorphism of the open set \( O_I \) onto \( \text{Prim} I \), whereas \( Q \mapsto p_I^{-1}(Q) \) is a homeomorphism of \( \text{Prim} \mathcal{A}/I \) onto the closed set \( h(I) \) of \( \text{Prim} \mathcal{A} \). A general study of the primitive ideal space of AF algebras was pursued in [2, 4, 9].

We collect some immediate properties of the primitive ideal space of \( \mathfrak{A} \) in the following
Remark 8. (i) For each $\theta \in I$, $\{I_\theta\} = \{I_\theta\}$.
(ii) For each $\theta \in \mathbb{Q}_{(0,1)}$, $I_\theta \notin I_\theta^+$, $I_\theta \notin I_\theta^+$, and $I_\theta = I_\theta^+ \cap I_\theta^-$. We also have $I_0 \notin I_0^+$ and $I_1 \notin I_1^-$. Therefore $\{I_0\} = \{I_0, I_0^+, I_0^-\}$ whenever $\theta \in \mathbb{Q}_{(0,1)}$, $\{I_0\} = \{I_0, I_0^+\}$ and $\{I_1\} = \{I_1, I_1^-\}$, showing in particular that the Jacobson topology on Prim $\mathfrak{A}$ is not Hausdorff. In spite of this we shall see that after removing the “singular points” $I_\theta^+$ from Prim $\mathfrak{A}$ we retrieve the usual topology on $[0,1]$.

For each set $E \subseteq [0,1]$, consider the ideal
\[ \mathfrak{I}(E) := \bigcap_{\theta \in E} I_\theta, \] (3.1)
and denote by $\overline{E}$ the usual closure of $E$ in $[0,1]$.

Lemma 9. $\mathfrak{I}(E) = \mathfrak{I}(\overline{E})$ for every set $E \subseteq [0,1]$.

Proof. The inclusion $\mathfrak{I}(\overline{E}) \subseteq \mathfrak{I}(E)$ is obvious by (3.1). We prove $\mathfrak{I}(E) \subseteq I_x$ for all $x \in \overline{E}$. Suppose ad absurdum there is $x \in \overline{E}$ for which $\mathfrak{I}(E) \notin I_x$, i.e. there is $(n,j) \in V$ with $(n,j) \in D(\mathfrak{I}(E))$ and $(n,j) \notin D(I_x)$. The latter and Remark 5 yield
\[ r(n,j-1) < x < r(n,j+1). \] (3.2)

On the other hand, because $D(\mathfrak{I}(E))$ contains $(n,j)$, every diagram $D(I_\theta)$, $\theta \in E$, must contain the whole “pyramid” starting at $(n,j)$, see Figure 13. Thus
\[ \forall \theta \in E, \forall k \geq 1, \theta \in [0,r(n+k,2^k j - 2^k + 1),1] \cup [r(n+k,2^k j + 2^k - 1),1]. \]

But
\[ r(n+k,2^k j + 2^k - 1) = \frac{kp(n,j+1) + p(n,j)}{kq(n,j+1) + q(n,j)} \xrightarrow{k \to \infty} \frac{p(n,j+1)}{q(n,j+1)} = r(n,j+1) \]
and
\[ r(n+k,2^k j - 2^k + 1) = \frac{kp(n,j-1) + p(n,j)}{kq(n,j-1) + q(n,j)} \xrightarrow{k \to \infty} \frac{p(n,j-1)}{q(n,j-1)} = r(n,j-1), \]

hence
\[ E \subseteq [0,r(n,j-1)] \cup [r(n,j+1),1], \]
which is in contradiction with (3.2). \hfill \Box

Figure 13. The ideal generated by $(n,j)$.  

\[ \begin{align*} 
(n,j-1) & \quad (n,j) & \quad (n,j+1) \\
(n+1,2j-2) & \quad (n+1,2j-1) & \quad (n+1,2j+1) & \quad (n+1,2j+2) \\
(n+1,4j-4) & \quad (n+1,4j-3) & \quad (n+1,4j+3) & \quad (n+1,4j+4) \\
\end{align*} \]
Remark 10. We have \( q(n, 2j) = q(n - 1, j) < \min\{q(n, 2j - 1), q(n, 2j + 1)\} \), so if \( r(n, 2j) = \frac{p}{q} \), then
\[
r(n, 2j + 1) - r(n, 2j - 1) = \frac{1}{q(n, 2j - 1)q(n, 2j)} + \frac{1}{q(n, 2j)q(n, 2j + 1)} < \frac{2}{q^2}.
\]
One can give a better estimate as follows. Let \( \theta = \frac{p}{q} \in (0, 1) \) be a rational number in lowest terms and let \( \bar{p} \in \{1, \ldots, q - 1\} \) denote the multiplicative inverse of \( p \) modulo \( q \). Let \( n_0 = n_0(\theta) \) be the smallest \( n \) such that \( \theta = r(n, j_0) \) for some \( j_0 \). Then \( j_0 \) is odd and the labels \( r' = \frac{p}{q} \) and respectively \( r'' = \frac{\bar{p}}{\bar{q}} \) of the “left parent” \( (n_0 - 1, q(n_0 - 1, j_0) \) and respectively of the “right parent” \( (n_0 - 1, \frac{q(n_0 - 1, j_0 - 1)}{2}) \) of the vertex \( (n_0, j_0) \), are given by \( (p', q') = (\bar{p}, \frac{p - q'}{q}) \), and respectively by \( (p'', q'') = (q - \bar{p}, q - q') = (q, q') \). Furthermore, we have \( r(n_0 + k, 2^k j_0 - 1) = \frac{\bar{p} + kp''}{q'\bar{q}''}, r(n_0 + k, 2^k j_0 + 1) = \frac{\bar{p} + kp' - q''}{q'\bar{q}''}, \) and
\[
\max\left\{ \frac{r(n_0 + k, 2^k j_0 + 1)}{r(n_0 + k, 2^k j_0)} \right\} < \frac{1}{kq''}.
\]

Lemma 11. For some \( x \in [0, 1] \) and \( S \subset [0, 1] \) suppose \( \mathcal{J}(S) \subseteq \mathcal{J}_x \). Then \( x \in \overline{S} \).

Proof. Obviously two cases may occur.

Case I: \( x \notin \mathbb{Q} \). Let \( (\frac{p}{q}) \) denote the sequence of continued fraction approximations of \( x \). Taking stock on the definition of the ideal \( \mathcal{J}_x \) we get positive integers \( k_1 < k_2 < \cdots \) and vertices \( (k_n, j_n) \in D(\mathfrak{A}) \) with the following properties:

(i) \( r(k_n, j_n) = \frac{p_n}{q_n} \);
(ii) \( j_n \) is even;
(iii) \( (k_n, j_n) \notin D(\mathcal{J}_x) \).

Actually (iii) is a plain consequence of (i) and gives in turn, cf. Remark 5,
\[
r(k_n, j_n - 1) < x < r(k_n, j_n + 1). \tag{3.3}
\]

Case II: \( x \in \mathbb{Q} \). There is \( n_0 \) such that \( (n, j_n) \notin D(\mathcal{J}_x) \) and \( r(n, j_n) = x \) for all \( n \geq n_0 \). In this case we take \( k_n = n \).

Suppose that \( \exists n \geq n_0, \forall \theta \in S, (k_n, j_n) \in D(\mathcal{J}_\theta) \). Then \( (k_n, j_n) \in D(\mathcal{J}(S)) \setminus D(\mathcal{J}_x) \), which contradicts the assumption of the lemma. Therefore we must have
\[
\forall n, \exists \theta_n \in S, (k_n, j_n) \notin D(\mathcal{J}_{\theta_n}),
\]
which according to Remark 5 gives
\[
r(k_n, j_n - 1) < \theta_n < r(k_n, j_n + 1). \tag{3.4}
\]

From (3.3), (3.4) and Remark 10 we now infer
\[
|x - \theta_n| < r(k_n, j_n + 1) - r(k_n, j_n - 1) < \frac{2}{q^2_n}, \quad \forall n \geq n_0,
\]
and so \( \text{dist}(x, S) = 0 \). This concludes the proof of the lemma.

As a consequence, the Jacobson topology is Hausdorff when restricted to the subset \( \text{Prim}_0 \mathfrak{A} = \{ I_\theta : \theta \in [0, 1] \} \) of \( \text{Prim} \mathfrak{A} \). Moreover, we have

Corollary 12. Let \( (\theta_n) \) be a sequence in \([0, 1]\). The following are equivalent:

(i) \( \theta_n \to \theta \) in \([0, 1]\).
(ii) \( I_{\theta_n} \to I_\theta \) in \( \text{Prim} \mathfrak{A} \).
Proof. (i) Suppose $\theta_n \to \theta$ in $[0, 1]$ but $I_{\theta_n} \not\to I_\theta$ in Prim $\mathfrak{A}$. Then there is $I$ ideal in $\mathfrak{A}$ such that $I \not\subset I_\theta$ and there is a subsequence $(n_k)$ such that $I_{\theta_{n_k}} \not\in \mathcal{O}_I$, so that $I \subset I_{\theta_{n_k}}$. By Lemma 9 this also yields $I \subset I_\theta$, which is a contradiction.

(ii) Suppose $I_{\theta_n} \to I_\theta$ in Prim $\mathfrak{A}$ but $\theta_n \not\to \theta$ in $[0, 1]$. Then there is a subsequence $(n_k)$ such that $\theta \not\in \{\theta_{n_k}\}_k$. By Lemma 11 we have $I := \bigcap_k I_{\theta_{n_k}} \not\subset I_\theta$, and so $I_\theta \in \mathcal{O}_I$. But on the other hand $I \subset I_{\theta_{n_k}}$, i.e. $I_{\theta_{n_k}} \not\in \mathcal{O}_I$ for all $k$, thus contradicting $I_{\theta_{n_k}} \to I_\theta$. \hfill $\square$

4. A DESCRIPTION OF THE DIMENSION GROUP

By a classical result of Elliott ([12], see also [11]), AF algebras are classified up to isomorphism by their dimension groups. In this section we give a description of the dimension group $K_0(\mathfrak{C})$ of the codimension one ideal $\mathfrak{C}$ of $\mathfrak{A}$ obtained by erasing all vertices labelled by $\frac{1}{2}$ from the Bratteli diagram. This is inspired by the generating function identity [6]

$$\sum_{n \geq 0} \theta_n X^n = \prod_{k \geq 0} (1 + X^{2^k} + X^{2^{k+1}}),$$

where $(\theta_n)_{n=0}^\infty$ is the Stern-Brocot sequence $q(0,0), q(1,0), q(1,1), q(2,0), q(2,1), q(2,2), q(2,3), \ldots, q(n,0), \ldots, q(n,2^n-1), q(n+1,0), \ldots$

For each integer $n \geq 0$, set

$$p_{n,k}(X) := \begin{cases} 1 & \text{if } k = 0, \\ X^k + X^{-k} & \text{if } 1 \leq k < 2^n, \end{cases}$$

and consider the abelian additive group

$$\mathcal{P}_n := \left\{ \sum_{0 \leq k < 2^n} c_k p_{n,k} : c_k \in \mathbb{Z} \right\}.$$

Set

$$\varrho(X) = X^{-1} + 1 + X, \quad \varrho_n(X) = \prod_{0 \leq k < n} \varrho(X^{2^k}),$$

and define the injective group morphisms

$$\beta_m : \mathcal{P}_m \to \mathcal{P}_{m+1}, \quad (\beta_m(p))(X) = \varrho(X)p(X^2),$$

$$\beta_{m,n} : \mathcal{P}_m \to \mathcal{P}_n, \quad (\beta_{m,n}p)(X) = (\beta_{m-1} \cdots \beta_m(p))(X) = \varrho_{m-n}(X)p(X^{2^{n-m}}), \quad m < n.$$

Note that

$$(\beta_m(p_{n,k}))(X) = \varrho(X)p_{n,k}(X)$$

$$= \begin{cases} p_{n+1,0}(X) + p_{n+1,1}(X) & \text{if } k = 0, \\ p_{n+1,2^{k-1}}(X) + p_{n+1,2k}(X) + p_{n+1,2k+1}(X) & \text{if } 1 \leq k < 2^n. \end{cases} \quad (4.1)$$

The group $K_0(\mathfrak{C}_n)$ identifies with the free abelian group $\mathbb{Z}^{2^n}$, generated by the Murray-von Neumann equivalence classes $[e_{n,k}]$ of minimal projections $e_{n,k}$ in the central summand $\mathfrak{A}_{(n,k)}$, $0 \leq k < 2^n$. We have $K_0(\mathfrak{C}) = \varprojlim K_0(\mathfrak{C}_n)$, the injective morphisms $\alpha_n : K_0(\mathfrak{C}_n) \to K_0(\mathfrak{C}_{n+1})$ being given by

$$\alpha_n([e_{(n,k)}]) = \begin{cases} [e_{(n+1,0)}] + [e_{(n+1,1)}] & \text{if } k = 0, \\ [e_{(n+1,2k-1)}] + [e_{(n+1,2k)}] + [e_{(n+1,2k+1)}] & \text{if } 1 \leq k < 2^n. \end{cases}$$
The positive cone $K_0(\mathfrak{C}_n)^+$ consists of elements of form $\sum_{k=0}^{2^n-1} c_k[e_{(n,k)}]$, $c_k \in \mathbb{Z}_+$. The groups $K_0(\mathfrak{C}_n)$ and $\mathcal{P}_n$ are identified by the group isomorphism $\phi_n$ mapping $[e_{(n,k)}]$ onto $p_{(n,k)}$. Equalities (4.1) are reflected into the commutativity of the diagram

$$
\begin{array}{ccc}
K_0(\mathfrak{C}_n) & \xrightarrow{\phi_n} & \mathcal{P}_n \\
\alpha_n \downarrow & & \beta_n \\
K_0(\mathfrak{C}_{n+1}) & \xrightarrow{\phi_{n+1}} & \mathcal{P}_{n+1}
\end{array}
$$

(4.2)

As a result, $K_0(\mathfrak{C})$ is isomorphic with the abelian group $\mathcal{P} = \lim(\mathcal{P}_n, \beta_n)$ and can, therefore, be described as $(\cup_n \mathcal{P}_n)/\sim = \mathbb{Z}[X + X^{-1}]/\sim$ where $\sim$ is the equivalence relation given by equality on each $\mathcal{P}_n \times \mathcal{P}_n$, and for $p \in \mathcal{P}_m$, $q \in \mathcal{P}_n$, $m < n$, by

$$p \sim q \iff q(X) = (\beta_{m,n}(p))(X) = p(X^{2n-m}) \prod_{0 \leq k < n-m} (X^{-2k} + 1 + X^{2k}).$$

Let $[p]$ denote the equivalence class of $p \in \cup_n \mathcal{P}_n$. The addition on $\mathcal{P}$ is given by

$$[p] + [q] = [\beta_{m,n}(p) + q], \quad p \in \mathcal{P}_m, \quad q \in \mathcal{P}_n, \quad m \leq n,$

and does not depend on the choice of $m$ or $n$. For example

$$[X^{-1} + X] + [X^{-3} + X^3] = [(X^{-1} + 1 + X)(X^{-2} + X^2) + X^{-3} + X^3]
= [2(X^{-3} + X^3) + (X^{-2} + X^2) + (X^{-1} + X)].$$

An element $[p]$, $p \in \mathcal{P}_n$, belongs to the positive cone $\mathcal{P}^+$ of the dimension group precisely when there is an integer $N > n$ such that $\beta_{n,N}(p)$ has nonnegative coefficients. The equality (where $c_{r+1} = 0$)

$$(X^{-1} + 1 + X) \sum_{0 \leq k < 2^n} c_k(X^{2k} + X^{-2k})$$

$$= \sum_{0 \leq k < 2^n} c_k(X^{2k} + X^{-2k}) + \sum_{0 \leq k < 2^n} (c_k + c_{k+1})(X^{2k+1} + X^{-2k-1})$$

shows that $p(X)$ has nonnegative coefficients if and only if $g(X)p(X^2)$ has the same property. Therefore $[p] \in \mathcal{P}^+$ precisely when $p(X)$ has nonnegative coefficients.

Consider the positive integers $q'_{(n,k)}$, $n \geq 0$, $0 \leq k < 2^n$, describing the sizes of central summands in

$$\mathfrak{C}_n = \bigoplus_{0 \leq k < 2^n} \mathbb{M} q'_{(n,k)},$$

(4.3)

that is

$$\begin{cases}
q'_{(n,0)} = q'_{(n,2^{n-1})} = 1,
q'_{(n,2k)} = q'_{(n-1,k)},
q'_{(n,2k+1)} = q'_{(n-1,k)} + q'_{(n-1,k+1)}, \quad 0 \leq k < 2^n.
\end{cases}$$

For instance $q'(3, k)$, $0 \leq k \leq 7$, are given by 1, 3, 2, 3, 1, 2, 1, 1, and $q'(4, k)$, $0 \leq k \leq 15$, by 1, 4, 3, 5, 2, 5, 3, 4, 1, 4, 3, 2, 3, 1, 2, 1, 1. From (4.3) we have

$$\sum_{0 \leq k < 2^n} q'_{(n,k)}[e_{(n,k)}] = [1] \quad \text{in } K_0(\mathfrak{C}).$$
This corresponds to
\[ \sum_{0 \leq k < 2^n} q'(n, k)p(n, k)(X) = g_n(X). \] (4.4)

One can give a representation of \( K_0(\mathcal{C}) \) where the injective maps \( \beta_n \) in (4.2) are replaced by inclusions \( \iota_n(p) = p \). Define
\[ \phi(n, k)(X) = \frac{p(n, k)(X^{1/2^n})}{g(n, k)(X^{1/2^n})} = \begin{cases} 1 & \text{if } k = 0, \\ \prod_{j=1}^{n-1} (X^{1/2^j} + 1 + X^{1/2^j}) & \text{if } 1 \leq k < 2^n, \end{cases} \]
and consider the additive abelian group
\[ \mathcal{R}_n := \left\{ \sum_{0 \leq k < 2^n} c_k\phi(n, k) : c_k \in \mathbb{Z} \right\}. \]

The equalities (4.1) become
\[ \begin{cases} \phi(n+1, 0) + \phi(n+1, 1) = \phi(n, 0), \\ \phi(n+1, 2k-1) + \phi(n+1, 2k) + \phi(n+1, 2k+1) = \phi(n, k), \end{cases} \quad 1 \leq k < 2^n, \]
and show that \( \mathcal{R}_n \subseteq \mathcal{R}_{n+1} \) and that the diagram
\[
\begin{array}{ccc}
K_0(\mathcal{C}_n) & \xrightarrow{\psi_n} & \mathcal{R}_n \\
\alpha_n \downarrow & & \downarrow \iota_n \\
K_0(\mathcal{C}_{n+1}) & \xrightarrow{\psi_{n+1}} & \mathcal{R}_{n+1}
\end{array}
\]
is commuting, where \( \psi([e(n, k)]) = \phi(n, k) \). Therefore \( K_0(\mathcal{C}) = \mathcal{R} := \cup_n \mathcal{R}_n \). Taking \( X = e^Y \), we see that \( K_0(\mathcal{C}) \) can be viewed as the \( \mathbb{Z} \)-linear span of \( \tilde{\phi}(n, k), n \geq 0, 0 \leq k < 2^n \), where
\[ \tilde{\phi}(n, k)(Y) = \begin{cases} 1 & \text{if } k = 0, \\ \prod_{j=1}^{n-1} (1 + 2 \cosh(Y/2^j)) & \text{if } 1 \leq k < 2^n, \end{cases} \]

One can certainly replace \( Y \) by \( iY \) and use \( \cos \) instead of \( \cosh \).

5. Traces on \( \mathfrak{A} \)

We augment the diagram \( \mathcal{G} = D(\mathfrak{A}) \) into \( \tilde{\mathcal{G}} \), by adding a \((-1)^{\text{st}}\) floor with only one vertex \( \ast = (-1, 0) \) connected to both \((0, 0)\) and \((0, 1)\). Traces \( \tau \) on \( \mathfrak{A} \) are in one-to-one correspondence (cf., e.g., [13, Section 3.6]) with families \( \alpha^\tau = (\alpha^\tau(n, k)) \) of numbers in \([0, 1], n \geq -1, 0 \leq k \leq 2^n \), such that
\[
\begin{cases}
\alpha^\tau = 1, \\
\alpha^\tau(n, 0) = \alpha^\tau(n+1, 0) + \alpha^\tau(n+1, 1) & \text{if } n \geq -1, \\
\alpha^\tau(n, 2^n) = \alpha^\tau(n+1, 2^n+1) + \alpha^\tau(n+1, 2^n+1) & \text{if } n \geq 0, \\
\alpha^\tau(n, k) = \alpha^\tau(n+1, 2k-1) + \alpha^\tau(n+1, 2k) + \alpha^\tau(n+1, 2k+1) & \text{if } n \geq 1, 0 < k < 2^n.
\end{cases}
\]
An inspection of $\tilde{\mathcal{G}}$ shows that such a family $\alpha^\tau$ is uniquely determined by the numbers $\alpha^\tau_{(n,k)}$ with odd $k$. Let $\mathcal{T}$ denote the diagram obtained by removing the memory in $\tilde{\mathcal{G}}$. Its set of vertices $V(\mathcal{T})$ consists of $\star$ and $(n, k)$ with $n \geq 0$ and odd $k$. For $v = (n, k)$ define $L_v = (n + 1, 2k - 1)$ if $n \geq 0$, $0 < k \leq 2^n$, and $R_v = (n, 2k + 1)$ if $n \geq -1$, $0 \leq k < 2^n$.

\[
\text{Figure 14. The diagram } \mathcal{T} \text{ in the dyadic representation.}
\]

Given $\alpha^\tau_v$, $v = (n, k) \in V(\mathcal{T})$, define recursively for $r \geq 1$

\[
\begin{align*}
\alpha^\tau_{(n+r,0)} &= \alpha^\tau_{(n+r-1,0)} - \alpha^\tau_{(n+r,1)} & \text{if } n \geq -1, \\
\alpha^\tau_{(n+r,2^{n+r})} &= \alpha^\tau_{(n+r-1,2^{n+r}-1)} - \alpha^\tau_{(n+r,2^{n+r}-1)} & \text{if } n \geq 0, \\
\alpha^\tau_{(n+r,2^r k)} &= \alpha^\tau_{(n+r-1,2^r-1 k)} - \alpha^\tau_{(n+r,2^r k-1)} - \alpha^\tau_{(n+r,2^r k+1)} & \text{if } n \geq 1,
\end{align*}
\]

or equivalently

\[
\begin{align*}
\alpha^\tau_{(n,0)} &= \alpha^\tau_* - \sum_{j=0}^{n} \alpha^\tau_{(j,1)} = \alpha^\tau_* - \sum_{j=0}^{n} \alpha^\tau_{L_j R_*} & \text{if } n \geq 0, \\
\alpha^\tau_{(n,2^n)} &= \alpha^\tau_{(0,1)} - \sum_{j=1}^{n} \alpha^\tau_{(j,2^{j}-1)} = \alpha^\tau_{(0,1)} - \sum_{j=1}^{n} \alpha^\tau_{R_{j-1} L_{(0,1)}} & \text{if } n \geq 1, \\
\alpha^\tau_{(n+r,2^r k)} &= \alpha^\tau_{(n,k)} - \sum_{j=1}^{r} \left( \alpha^\tau_{(n+j,2^j k-1)} + \alpha^\tau_{(n+j,2^j k+1)} \right) \\
&= \alpha^\tau_{(n,k)} - \sum_{j=1}^{r} \left( \alpha^\tau_{R_{j-1} L_{(n,k)}} + \alpha^\tau_{L_j R_{(n,k)}} \right) & \text{if } n \geq 2.
\end{align*}
\]

(5.1)

There is an obvious order relation on $V(\mathcal{T})$ defined by $(n, k_n) \preceq (n', k'_n)$ if $n \leq n'$ and there is a chain of vertices $(n, k_n), \ldots, (n', k'_n)$ such that $(n + i, k_{n+i})$ is connected to $(n + i + 1, k_{n+i+1})$, i.e. $k_{n+i+1} - 2k_{n+i} = \pm 1$. A function $f : V(\mathcal{T}) \to \mathbb{R}$ is monotonically decreasing if $f(v_1) \geq f(v_2)$ whenever $v_1 \preceq v_2$ in $V(\mathcal{T})$. For each vertex
\[ v = (n, k) \in V(T), \text{ let} \]
\[
C_v = \begin{cases} 
\{L^j R^* : j \geq 0\} & \text{if } v = \star, \\
\{R^{-1} L(0, 1) : j \geq 1\} & \text{if } v = (0, 1), \\
\{R^{-1} L_v : j \geq 1\} \cup \{L^j R_v : j \geq 1\} & \text{if } v \in V(T) \setminus \{\star, (0, 1)\},
\end{cases} \tag{5.2}
\]
denote the set of vertices in \( V(T) \) neighboring the vertical infinite segment originating at \( v \). As a result of (5.1) and of non-negativity of \( \alpha^T \) we have

**Proposition 13.** There is a one-to-one correspondence between traces on \( \mathfrak{A} \) and functions \( \phi : V(T) \to [0, 1] \) such that \( \phi(\star) = 1 \) and
\[
\phi(v) \geq \sum_{w \in C_v} \phi(w), \quad \forall v \in V(T). \tag{5.3}
\]

Note that a function satisfying (5.3) is necessarily monotonically decreasing.

One can give a description of the set \( C_v \) using the one-to-one correspondence \( v \mapsto r(v) \) between the sets \( V(T) \) and \( \mathbb{Q} \cap [0, 1] \) (see Figure 15). Any number in \( \mathbb{Q} \cap (0, 1) \) can be uniquely represented as a (reduced) continued fraction \([a_1, \ldots, a_t]\) with \( a_t \geq 2 \). It is not hard to notice and prove that, for any \( v \in V(T) \) with \( r(v) = [a_1, \ldots, a_t] \), \( a_t \geq 2 \), we have
\[
r(Lv) = \begin{cases} 
[a_1, \ldots, a_{t-1}, a_t - 1, 2] & \text{if } t \text{ even}, \\
[a_1, \ldots, a_{t-1}, a_t + 1] & \text{if } t \text{ odd},
\end{cases}
\]
\[
r(Rv) = \begin{cases} 
[a_1, \ldots, a_{t-1}, a_t + 1] & \text{if } t \text{ even}, \\
[a_1, \ldots, a_{t-1}, a_t - 1, 2] & \text{if } t \text{ odd}.
\end{cases} \tag{5.4}
\]
As a result of (5.2) and (5.4) we have
\[
\{r(w) : w \in C_v\} = \{[a_1, \ldots, a_{t-1}, a_t - 1, 1, k] : k \geq 1\} \cup \{[a_1, \ldots, a_{t-1}, a_t + k : k \geq 1]\},
\]

**Figure 15.** The diagram \( T \) in the continued fraction representation.
which shows in conjunction with Proposition 13 that there is a one-to-one correspondence between traces on \( \mathfrak{A} \) and maps \( \phi : \mathbb{Q} \cap [0,1] \to [0,1] \) which satisfy

\[
\begin{align*}
1 = \phi(0) & \geq \sum_{k=1}^{\infty} \phi\left(\frac{1}{k}\right), \\
\phi(1) & \geq \sum_{k=1}^{\infty} \phi\left(\frac{k}{k+1}\right), \\
\phi([a_1, \ldots, a_t]) & \geq \sum_{k=1}^{\infty} \left(\phi([a_1, \ldots, a_{t-1}, a_t - 1, 1, k]) + \phi([a_1, \ldots, a_{t-1}, a_t + k])\right).
\end{align*}
\]

6. Generators, relations, and braiding

We shall use the path model for AF algebras as in [15, Section 2.3.11] and [13, Section 2.9]. Here however a monotone increasing path \( \xi \) will be encoded by the sequence \( (\xi_n) \) where \( \xi_n \) gives the “horizontal coordinate” of the vertex at floor \( n \), instead of its edges. To use this model we again augment the diagram \( \mathcal{G} = D(\mathfrak{A}) \) into \( \tilde{\mathcal{G}} \).

Denote by \( \Omega \) the (uncountable) set of monotone increasing paths starting at \( \star \). Let \( \Omega_r \), denote the set of infinite monotone increasing paths starting on the \( r \)-th floor of \( \tilde{\mathcal{G}} \), \( \Omega_{[r,s]} \) the set of monotone increasing paths that connect \( \star \) with a vertex on the \( r \)-th floor, and \( \Omega_{[r,s]} \) the set of monotone increasing paths starting on the \( r \)-th floor and ending on the \( s \)-th floor. Let \( \xi \) \( \in \Omega_r \), \( \xi_{[r,s]} \in \Omega_{[r,s]} \), \( \xi \in \Omega_s \) denote the natural truncations of a path \( \xi \in \Omega \). By \( \xi \circ \eta \) we denote the natural concatenation of two paths \( \xi \in \Omega_{[r]} \) and \( \eta \in \Omega_s \) with \( \xi_r = \eta_r \). Consider the set \( R_r \) of pairs of paths \( (\xi, \eta) \in \Omega_{[r]} \times \Omega_{[r]} \) with the same endpoint \( \xi_r = \eta_r \). For each \( (\xi, \eta) \in R_r \), the mapping

\[
\Omega \ni \omega \mapsto T_{\xi,\eta} \omega = \delta(\eta, \omega_r) \xi \circ \omega_{[r]} \in \Omega,
\]

extends to a linear operator on the \( \mathbb{C} \)-linear space \( \mathbb{C} \Omega \) with basis \( \Omega \), and also to a bounded operator \( T_{\xi,\eta} : \ell^2(\Omega) \to \ell^2(\Omega) \) with \( \|T_{\xi,\eta}\| = 1 \). We have \( \mathfrak{A} = \bigcup_{r \geq 1} \mathfrak{A}_r \), where the linear span \( \mathfrak{A}_r \) of the operators \( T_{\xi,\eta} \), \( (\xi, \eta) \in R_r \), forms a finite dimensional \( C^* \)-algebra as a result of

\[
T_{\eta,\xi}^* = T_{\xi,\eta}, \quad T_{\xi,\eta} T_{\xi',\eta'} = \delta(\eta, \xi') T_{\xi,\eta'}, \quad \sum_{\xi \in \Omega_{[r]}} T_{\xi,\xi} = 1.
\]

Furthermore the inclusion \( \mathfrak{A}_r \xrightarrow{\iota_r} \mathfrak{A}_{r+1} \) is given by

\[
\iota_r(T_{\xi,\eta}) = \sum_{(\lambda, \zeta) \in \Omega_{[r+1]} \times \Omega_{[r+1]}} T_{\xi \circ \lambda, \zeta \circ \lambda}.
\]

This model is employed to give a presentation by generators and relations of the \( C^* \)-algebra \( \mathfrak{A} \) in the spirit of the presentation of the GICAR algebra from [13, Example 2.23]. We also construct two families of projections that satisfy commutation relations reminiscent of the Temperley-Lieb relations.

We consider the following elements in \( \mathfrak{A} \):

1. the projection \( e_n \) in \( \mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n \) onto the linear space of edges from N (north) to SW (south-west), \( n \geq 1 \).
2. the projection \( f_n \) in \( \mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n \) onto the linear space of edges from N to SE, \( n \geq 0 \).
3. the projection \( g_n = 1 - e_n - f_n \) in \( \mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n \) onto the linear space of edges from N to S, \( n \geq 0 \).
(4) the partial isometry $v_n \in A_{n-1,n+1} \subseteq A_{n+1}$ with initial support $v^*_n v_n = \tilde{e}_n = g_n f_{n+1}$ and final support $v_n^* v_n = f_n = f_n e_{n+1}$, which flips paths in the diamonds of shape N-S-SE-NE, $n \geq 0$.

(5) the partial isometry $w_n \in A_{n-1,n+2} \subseteq A_{n+1}$ with initial support $w^*_n w_n = \tilde{e}'_n = g_n e_{n+1}$ and final support $w_n w^*_n = f'_n = e_n f_{n+1}$, which flips paths in the diamonds of shape N-S-SW-NW, $n \geq 1$.

The AF-algebra $\mathfrak{A}$ is plainly generated by the set $\mathfrak{G} = \{e_n\}_{n \geq 1} \cup \{f_n\}_{n \geq 0} \cup \{v_n\}_{n \geq 0} \cup \{w_n\}_{n \geq 1}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{The generators of $\mathfrak{A}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17.png}
\caption{Support of projection $e_n$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure18.png}
\caption{Support of projection $f_n$.}
\end{figure}

Straightforward commutation relations arise since elements defined by edges that reach up to floor $\leq r$ commute with elements defined by edges between the $r^{th}$ and the $s^{th}$ floors with $r < s$, as a result of $[\mathfrak{A}_r, \mathfrak{A}'_r \cap \mathfrak{A}_s] = 0$. For instance $v_s$ commutes with
\( e_r, f_r, g_r \) if \( r \leq s - 1 \) or \( r \geq s + 2 \), and \([v_s, v_r] = [v_s, v^*_r] = [v_s, w_r] = [v_s, w^*_r] = 0 \) if \(|r - s| \geq 2\). Besides, the elements of \( \mathfrak{G} \) satisfy the following commutation relations:

\begin{enumerate}
\item[(R1)] \( e^2_n = e_n \), \( f^2_n = f^*_n \), \( g^2_n = g^*_n \), \( e_n + f_n + g_n = 1 \); \( e_n, f_n, g_n \) mutually commute.
\item[(R2)] \( (1 - f_n)v_n = (1 - e_n)v_n = 0 \), \( v_n(1 - g_n) = v_n(1 - f_{n+1}) = 0 \).
\item[(R3)] \( v_ng_n = f_n v_n, v_n f_{n+1} = e_n v_n, w_n g_n = e_n w_n, w_n e_{n+1} = f_{n+1} w_n \).
\end{enumerate}

As a result of (R3) we also get

\[
\begin{align*}
v_{n+1}v_n &= v^2_n = v_{n+1}v^*_n = v^*_n v_{n+1} = 0, \\
w_{n+1}w_n &= w^2_n = w_{n+1}w^*_n = w^*_n w_{n+1} = 0, \\
v_nw_n &= v_{n+1}w_n = w_{n+1}v_n = w_{n+1}v_{n+1} = 0, \\
v_nw^*_n &= v_{n+1}w^*_n = v^*_n w_n = v^*_n w_{n+1} = 0.
\end{align*}
\]

The only non-zero products \( ab \) with \( a \in \{v_n, v^*_n, w_n, w^*_n\} \) and \( b \in \{v_{n+1}, v^*_{n+1}, w_{n+1}, w^*_{n+1}\} \) are \( v_nv_{n+1}, w_nw_{n+1}, w^*_nv_{n+1}, \) and \( v^*_nw_{n+1} \).

Let \( B_n \) denote Artin's braid group generated by \( \sigma_1, \ldots, \sigma_{n-1} \) with relations \( \sigma_i \sigma_j = \sigma_j \sigma_i \) if \(|i - j| > 1\) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \). Relations (6.1) show in particular that, for every \( n \), the mappings \( \sigma_i \mapsto v_{i-1} \) and \( \sigma_i \mapsto w_i \) define representations of \( B_n \) on the algebra \( \mathfrak{A} \).

Taking \( R_n(\lambda) := 1 + \lambda v_n \), the equalities

\[
v^2_n = 0, \quad v_nv_{n+1} = 0
\]

yield the Yang-Baxter type relation

\[
R_n(\lambda)R_{n+1}(\lambda + \mu)R_n(\mu) = R_{n+1}(\mu)R_n(\lambda + \mu)R_{n+1}(\lambda).
\]

By analogy with the construction of Temperley-Lieb-Jones projections in the GICAR algebra (cf., e.g., [13] or [15]) for each \( \lambda > 0 \) we put \( \tau = \frac{\lambda}{(1 + \lambda)^2} \in [0, \frac{1}{4}] \) and consider

\[
E_n = \frac{1}{1 + \lambda} \left( v^*_n v_n + \sqrt{\lambda} v_n + \sqrt{\lambda} v^*_n + \lambda v_nv^*_n \right) \in \mathfrak{A}, \quad n \geq 0.
\]
Proof. The elements $E_n$ and $F_n$ define (self-adjoint) projections in the AF algebra $\mathfrak{A}$ satisfying the braiding relations

$$E_nF_n = F_nE_n = 0,$$  \hspace{1cm} (6.6)

$$[E_n, E_m] = [F_n, F_m] = [E_n, F_m] = 0 \text{ if } |n - m| \geq 2,$$  \hspace{1cm} (6.7)

$$E_nE_{n+1}E_n + \tau E_ne_{n+2}, \hspace{1cm} E_{n+1}E_nE_{n+1} = \tau E_{n+1}E_n,$$  \hspace{1cm} (6.8)

$$F_nF_{n+1}F_n = \tau F_nf_{n+2}, \hspace{1cm} F_{n+1}F_nF_{n+1} = \tau F_{n+1}F_n,$$  \hspace{1cm} (6.9)

$$E_nF_{n+1}E_n = \lambda \tau E_nf_{n+2}, \hspace{1cm} F_nE_{n+1}F_n = \lambda \tau F_{n+1}e_{n+2},$$  \hspace{1cm} (6.10)

$$E_{n+1}F_nE_{n+1} = \lambda \tau E_{n+1}e_n, \hspace{1cm} F_{n+1}E_nF_{n+1} = \lambda \tau F_{n+1}f_n,$$  \hspace{1cm} (6.11)

$$E_nE_{n+1}F_n = E_nF_{n+1}F_n = E_{n+1}E_nF_{n+1} = E_{n+1}F_nF_{n+1} = 0,$$  \hspace{1cm} (6.12)

$$F_nE_{n+1}E_n = F_nF_{n+1}E_n = F_{n+1}E_nE_{n+1} = F_{n+1}F_nE_{n+1} = 0.$$  \hspace{1cm} (6.13)

**Proposition 14.** The elements $E_n$ and $F_n$ define (self-adjoint) projections in the AF algebra $\mathfrak{A}$ satisfying the braiding relations

$$E_nF_n = F_nE_n = 0,$$  \hspace{1cm} (6.6)

$$[E_n, E_m] = [F_n, F_m] = [E_n, F_m] = 0 \text{ if } |n - m| \geq 2,$$  \hspace{1cm} (6.7)

$$E_nE_{n+1}E_n + \tau E_ne_{n+2}, \hspace{1cm} E_{n+1}E_nE_{n+1} = \tau E_{n+1}E_n,$$  \hspace{1cm} (6.8)

$$F_nF_{n+1}F_n = \tau F_nf_{n+2}, \hspace{1cm} F_{n+1}F_nF_{n+1} = \tau F_{n+1}F_n,$$  \hspace{1cm} (6.9)

$$E_nF_{n+1}E_n = \lambda \tau E_nf_{n+2}, \hspace{1cm} F_nE_{n+1}F_n = \lambda \tau F_{n+1}e_{n+2},$$  \hspace{1cm} (6.10)

$$E_{n+1}F_nE_{n+1} = \lambda \tau E_{n+1}e_n, \hspace{1cm} F_{n+1}E_nF_{n+1} = \lambda \tau F_{n+1}f_n,$$  \hspace{1cm} (6.11)

$$E_nE_{n+1}F_n = E_nF_{n+1}F_n = E_{n+1}E_nF_{n+1} = E_{n+1}F_nF_{n+1} = 0,$$  \hspace{1cm} (6.12)

$$F_nE_{n+1}E_n = F_nF_{n+1}E_n = F_{n+1}E_nE_{n+1} = F_{n+1}F_nE_{n+1} = 0.$$  \hspace{1cm} (6.13)

**Proof.** The initial and final projections of the partial isometry $v_n$ are orthogonal, thus $E_n$ defines a projection in $\mathfrak{A}_n$ for every $\lambda \geq 0$. A similar property holds for $F_n$, which is seen to be orthogonal to $E_n$. The commutation relations (6.7) are obvious because $v_{n+2}$ and $w_{n+2}$ commute with all elements in $\mathfrak{A}_{n+1}$, including $E_n$ and $F_n$. By (6.1) we have $v_n^*v_{n+1} = v_n^*v_{n+1} = 0$, leading to

$$E_nE_{n+1} = \frac{\sqrt{\lambda}}{(1 + \lambda)^2} \left( v_{n+1}^*v_n + \sqrt{\lambda} v_n + v_{n+1}^* v_{n+1}^* \right),$$  \hspace{1cm} (6.14)

and also

$$E_{n+1}E_n = (E_nE_{n+1})^* = \frac{\sqrt{\lambda}}{(1 + \lambda)^2} \left( v_{n+1}^* + \sqrt{\lambda} v_{n+1} v_{n+1}^* \right) \left( v_n^*v_n + \sqrt{\lambda} v_n^* \right).$$  \hspace{1cm} (6.15)
From (6.14) and $v_{n+1}E_n = v_n^*v_n = 0$ we have
\[
E_nE_{n+1} = \frac{\lambda}{(1 + \lambda)^3} (v_n^*v_n + \sqrt{\lambda} v_n) v_{n+1}(v_n^*v_n + \sqrt{\lambda} v_n^*) \tag{6.16}
\]
and
\[
E_{n+1}E_n = \frac{\lambda}{(1 + \lambda)^3} (\bar{e}_n + \sqrt{\lambda} \bar{f}_{n+1})(\bar{e}_n + \sqrt{\lambda} \bar{f}_{n+1}).
\]

But $\bar{e}_n\bar{f}_{n+1} = \bar{e}_n\bar{f}_{n+1} = g_n f_{n+1} e_{n+1} = v_n\bar{e}_n e_{n+1} e_{n+2} = v_n e_{n+2}$
and because $[e_{n+2}, v_n] = 0$ we have $\bar{e}_n\bar{f}_{n+2} = v_n\bar{e}_n e_{n+2} = v_n e_{n+2}$
and $\bar{f}_{n+1} = v_n f_{n+1} = v_n f_{n+1} v_n^* v_n e_{n+2} = v_n e_{n+2}$
which we insert in (6.16) to get
\[
E_nE_{n+1} = \tau E_n e_{n+2}.
\]

From (6.15) and $v_n^*E_{n+1} = v_n^*v_n^* = 0$ we find
\[
E_{n+1}E_n = \frac{\lambda}{(1 + \lambda)^3} (v_n^* v_{n+1} + \sqrt{\lambda} \tilde{f}_{n+1}) \bar{e}_n (v_{n+1} + \sqrt{\lambda} \tilde{f}_{n+1}). \tag{6.17}
\]

As a result of $[g_n, v_{n+1}] = 0$ and $(1 - f_{n+1}) v_{n+1} = 0$ we have $v_{n+1} \bar{e}_n v_{n+1} = \bar{e}_n v_{n+1}$.

It is also clear that $f_{n+1} e_{n+1} = f_{n+1} e_{n+1} = f_{n+1} g_{n+1} e_{n+1} = f_{n+1} g_{n+1} v_{n+1} = f_{n+1} g_{n+1} v_{n+1} = v_{n+1} g_{n+1}$
and $v_{n+1} \bar{e}_{n+1} = v_{n+1} g_{n+1} = v_{n+1} g_{n+1}$. Together with (6.17) these equalities yield
\[
E_{n+1}E_n = \tau E_{n+1} g_{n+1}.
\]

Equalities (6.9)–(6.12) are checked in a similar way. (6.13) follows by taking adjoints in (6.12).

\[\square\]

**Acknowledgments**

I am grateful to Ola Bratteli, Marius Dadarlat, George Elliott, Andreas Knauf, and Bruce Reznick for useful comments and suggestions.

**References**


Department of Mathematics, University of Illinois, 1409 W. Green Street, Urbana, IL 61801, USA

Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail: fboca@math.uiuc.edu