Print name

Math 418, Final Exam, Part I

May 12, 2014

No calculators or other e-devices and no books or notes are allowed.
Solve ALL problems. Time: 80 min.

[1] [10 pts.] Let $m, n \in \mathbb{N}$ and $d = \gcd(m, n) \in \mathbb{N}$. Show that

$$\gcd(X^m - 1, X^n - 1) = X^d - 1 \quad \text{in } \mathbb{Z}[X].$$

[2] [12 pts.] Let x_1, x_2, x_3 denote the roots of the polynomial $X^3 + aX^2 + bX + c$.

(i) Express the symmetric polynomial $p_5 = x_1^5 + x_2^5 + x_3^5$ as a function in a, b, c.

(ii) Do the same thing for the symmetric function

$$D = (x_1 - x_2)^2 (x_2 - x_3)^2 (x_1 - x_3)^2.$$

[3] [15 pts.] Suppose that F is a field and $p(X) \in F[X]$ is a monic polynomial with $\deg(p) > 0$. Prove that the following three statements are equivalent:

(i) $p(X)$ is irreducible in $F[X]$.

(ii) The quotient ring $F[X]/(p(X))$ is an integral domain.

(iii) $F[X]/(p(X))$ is a field.

[4] [13=4+3+6 pts.] (i) Let R be a ring, M be a right R-module and N be a left R-module. What is the definition of the tensor product of M and N over R (viewed as a $\mathbb{Z}$-module)?

(ii) Sketch the construction of the tensor product.

(iii) Prove the following isomorphisms:

(iii.1) $F \underset{R}{\otimes} M = 0$ if F is a field, R a subring of F and M a torsion module over R.

(iii.2) $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \cong \mathbb{Z}_d$ where $m, n \in \mathbb{N}$, $d = \gcd(m, n) \in \mathbb{N}$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

(iii.3) $F^m \underset{F}{\otimes} F^n \cong F^{mn}$ where F is a field.

Perfect score: 50 **points.**

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Math 418, Final Exam, Part II

May 12, 2014

No calculators or other e-devices. Books and lecture notes are allowed.
Solve ALL FIVE problems. Time: 80 min.

[1] [9 pts.] Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a squarefree integer greater than 3. Prove that 2, $\sqrt{-n}$ and $1 + \sqrt{-n}$ are irreducible in R.

[2] [10 pts.] Consider the polynomial

$$p(X) = X^3 - 6X^2 + 9X + 3.$$

(i) Why is $p(X)$ irreducible in $\mathbb{Q}[X]$?
(ii) Consider the extension $\mathbb{Q}(u)$ of $\mathbb{Q}$ generated by a real root u of $p(X)$. Express $3u^5 - u^4 + 2$ in terms of the basis $\{1, u, u^2\}$ of $\mathbb{Q}(u)$.

[3] [9 pts.] (i) Show that in the field $\mathbb{C}$ the subfields $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are isomorphic as $\mathbb{Q}$-vector spaces but not as fields.
(ii) If $n > 2$ and μ is a primitive n^{th} root of unity, find $[\mathbb{Q}(\mu + \mu^{-1}) : \mathbb{Q}]$.
(iii) Which roots of unity are contained in the fields $\mathbb{Q}(\sqrt{5})$ and respectively $\mathbb{Q}(\sqrt{-3})$?

[4] [10 pts.] (i) If $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, find $[F : \mathbb{Q}]$ and a basis of F over $\mathbb{Q}$.
(ii) Do the same for $F = \mathbb{Q}(i, \sqrt{3}, \omega)$, where $i \in \mathbb{C}$, $i^2 = -1$, and $\omega \neq 1$ is a cubic root of unity.

[5] [12 pts.] (i) Determine all possible rational canonical forms for a linear transformation with characteristic polynomial $X^2(X^2 + 1)^2$.
(ii) Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(X - 2)^3(X - 3)^2$.

Perfect score: 50 **points.**