Print name

Math 418, Hour Exam # 3, Time: 120 min.

April 30, 2014

No calculators or other e-devices and no books or notes are allowed.
Solve ALL SIX problems. All problems have the same weight.

[1] Let $\mathcal{C}$ denote the set of *constructible* real numbers.[1] Show that:
 (i) $\mathbb{Q} \subseteq \mathcal{C}$.
 (ii) If $a \in \mathcal{C}$ and $a \geq 0$, then $\sqrt{a} \in \mathcal{C}$.

[2] Let x_1, x_2, x_3 denote the roots of the polynomial $X^3 + aX^2 + bX + c$.
 (i) Express the discriminant

$$D = (x_1 - x_2)^2(x_2 - x_3)^2(x_1 - x_3)^2$$

as a function in a, b, c.
 (ii) Do the same thing for $p_5 = x_1^5 + x_2^5 + x_3^5$.

[3] (i) State the Fundamental Theorem of Symmetric Polynomials.
 (ii) Compute the highest term $\mathrm{ht}(f)$ (with respect to the lexicographic order induced on monomials) for

$$f(X_1, X_2, \ldots, X_n) = s_1^2 s_2 s_3 \cdots s_n,$$

where $s_i = s_i(X_1, X_2, \ldots, X_n)$ are the elementary symmetric polynomials in n variables.

[4] Let R be an integral domain, and let $T(M) = \{m \in M : \exists r \in R, r \neq 0, rm = 0\}$ denote the torsion submodule of an R-module M.
 (i) Show that $T\big(M/T(M)\big) = 0$.
 (ii) Show that $\varphi(T(M)) \subseteq T(N)$ for any $\varphi \in \mathrm{Hom}_R(M, N)$.

[5] (i) Let R be a unital commutative ring and M be a free R-module of finite rank n. Describe an explicit isomorphism between the R-algebras $\mathrm{End}_R(M)$ and $M_n(R)$ and prove its properties.
 (ii) What is a projective module? Show that any finitely generated free module is projective.

[6] (i) Define the (internal) direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ of the R-modules $M_1, M_2, \ldots, M_k$.
 (ii) State the Invariant Factors Theorem.
 (iii) Given a field F and a monic polynomial $a(X) \in F[X]$, $\deg(a) \geq 1$, describe the action of X on the quotient $F[X]/(a)$ viewed as a F-vector space. Compute the characteristic polynomial of the associated F-linear transformation.

Perfect score: $60 = 6 \times 10$ **points.**

[1] $a \in \mathbb{R}$ is constructible if a line segment of length $|a|$ can be constructed from a segment of length 1 in a finite number of steps using the compass and the straightedge