Print name

Math 418, Hour Exam # 1, Time: 60 min.

March 3, 2014

No calculators or other e-devices and no books or notes are allowed.
Solve ALL FIVE problems. Show complete work to qualify for full credit.
Problems carry the same weight.

[1] Let D be a squarefree integer, $D \neq 0, 1$, and let $\mathcal{O} = \mathcal{O}_{\sqrt{D}}$ denote the ring of integers in the field $\mathbb{Q}(\sqrt{D})$, endowed with the usual norm

$$N(a + b\omega) = (a + b\omega)(a - b\omega), a, b \in \mathbb{Q}, \quad \text{where } \omega = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4). \end{cases}$$

Using the customary properties of the norm N:

(i) Show that

$$\mathcal{O}^{\times} = \{\alpha \in \mathcal{O} : N(\alpha) = \pm 1\}.$$

(ii) Find $\mathcal{O}^{\times}$ when $\mathcal{O} = \mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-3})}$.

(iii) Sketch an argument showing that the set $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{D})}$ is finite when $D < 0$.

[2] (i) Give the definitions of the following classes of rings: Euclidean Domains, Principal Ideal Domains, Unique Factorization Domains.

(ii) Provide an explicit list of all irreducible elements of the ring $\mathbb{Z}[i]$. State (without proof) an important number theoretical consequence of this.

[3] Prove that if the polynomials $f(X), g(X) \in \mathbb{Z}[X]$ are both primitive, then their product $h(X) = f(X)g(X)$ is also primitive. [Recall that a polynomial $p(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$ is called *primitive* if $\gcd(a_0, a_1, \ldots, a_n) = 1$ in $\mathbb{Z}$.]

[4] Prove that for every prime integer p the polynomial

$$\Phi_p(X) = \sum_{k=0}^{p-1} X^k$$

is irreducible in $\mathbb{Z}[X]$. Is it also irreducible in $\mathbb{Q}[X]$?

[5] (i) Decompose the polynomial $f(X) = X^8 + 1$ as a product of irreducible factors in $\mathbb{R}[X]$.

(ii) Show that $f(X)$ is irreducible in $\mathbb{Z}[X]$.

Perfect score: $50 = 5 \times 10$ **points.**