1. Introduction

The principal aim of model theory is to create formal theories expressing properties of various mathematical structures, classify them according to whether they are "tame" or "wild", and study the properties of sets which are first order definable in mathematical structures with tame theories.

The two prototypical structures with tame theories are the field $(\mathbb{C}, +, \cdot, 0, 1)$ of complex numbers and the ordered field $(\mathbb{R}, <, +, \cdot, 0, 1)$ of real numbers. Tarski showed that any subset of $\mathbb{C}^n$ definable in $(\mathbb{C}, +, \cdot, 0, 1)$ is a boolean combination of subvarieties of $\mathbb{C}^n$ (i.e., sets carved out by polynomial equations) and any subset of $\mathbb{R}^n$ definable in $(\mathbb{R}, <, +, \cdot, 0, 1)$ is **semialgebraic**: a boolean combination of solution sets of polynomial inequalities. The description of $(\mathbb{C}, +, \cdot, 0, 1)$ definable sets is equivalent to Chevally's theorem on constructible sets from algebraic geometry. The description of $(\mathbb{R}, <, +, \cdot, 0, 1)$ definable sets is of fundamental importance to the geometry of semialgebraic sets, for example it is used to show that any semialgebraic set is triangulable. The canonical wild structure is the ring $(\mathbb{Z}, +, \cdot, 0, 1)$ of integers. The proof of Godel's incompleteness theorem shows that virtually any subset of $\mathbb{Z}^n$ encountered by a typical mathematician is $(\mathbb{Z}, +, \cdot, 0, 1)$-definable.

A survey of natural mathematical structures whose theories are understood reveals a deep empirical dichotomy: the structure either defines an isomorphic copy of $(\mathbb{Z}, +, \cdot, 0, 1)$ (or something close) or every definable set admits a simple description. This dichotomy is a foundational manifestation of the principal role of $(\mathbb{Z}, +, \cdot, 0, 1)$ in mathematics.

I discuss two ongoing projects that I believe to be my strongest. The first is a joint project with Philipp Hieronymi on the geometry and topology of subsets of $\mathbb{R}^n$ definable in first order expansions of $(\mathbb{R}, <, +, 0)$. We have shown that expansions which do not define a copy of $(\mathbb{Z}, +, \cdot, 0, 1)$ fall into one of two restrictive classes, one related to semialgebraic geometry and one related to automata theory. The second project is the interpolative fusion program, joint with Minh Chieu Tran and Alex Kruckman. The first project explores the boundary between tame and wild behavior in a specific setting, the second provides a higher level perspective on tame theories of interest.

Subsection 2.3 describes a possible model-theoretic approach to an old graph-theoretic conjecture of Seese. This conjecture is related to questions arising from my work with Philipp.

1.1. Expansions of $(\mathbb{R}, <, +, 0)$.

Philipp and I developed a classification of first order expansions of $(\mathbb{R}, <, +, 0)$ according to the geometry and topology of their definable sets. The classification consists of three types: A, B, and C. An expansion which defines every compact subset of every $\mathbb{R}^n$ is **type C**. An expansion is **type A** if every definable subset X of $\mathbb{R}$ that admits a definable ordering with the same order type as $(\mathbb{N}, <)$, is necessarily nowhere dense. For example, the ordered field of real numbers is type A as every countable semialgebraic set is finite.

The geometry of definable sets in type A expansions is the ultimate generalization of semialgebraic geometry in the setting of first order expansions of the ordered group $(\mathbb{R}, <, +, 0)$ of real numbers. Definable sets in type A expansions enjoy natural weakenings of properties of semialgebraic sets. Many basic results on semialgebraic sets may be obtained by specializing general results on type A expansions. For example a type A expansion cannot define a continuous surjection $[0, 1]^n \to [0, 1]^m$ when $n < m$.

An expansion is **type B** if it is not type A or type C. This definition is justified by its strong consequences, such as Theorem 2.8 below (which asserts that continuous functions definable in type B expansions are weakly periodic in a certain sense). Type B expansions are connected to the monadic second order theory of $(\mathbb{N}, <)$, known as **S1S**. [1] This theory has been extensively studied in connection with automata theory beginning with Büchi's theorem [4] that S1S is decidable. Büchi's proof relies on a correspondence between formulas of S1S and finite automata.

There is a body of work by computer scientists and automata theorists on specific examples of type B expansions of $(\mathbb{R}, <, +, 0)$. The theories of these examples are bi-interpretable with S1S, which results in correspondences between formulas and automata. The work in [3] describes a type B expansion $\mathfrak{R}$ of $(\mathbb{R}, <, +, 0)$ and a way of associating a linear iterated function system to a finite automata in such a way that a subset of $\mathbb{R}^n$ defined over $\mathfrak{R}$ by a formula φ is the attractor (in the sense of fractal geometry) of the iterated function system associated to the automaton corresponding to φ. This provides a nexus between model theory, automata theory, and fractal geometry.

One contribution of our work is the realization that these specific structures fall into a natural general class of expansions of $(\mathbb{R}, <, +, 0)$. A study of these expansions should produce new results in automata theory and show that known results on automata (such as Cobham's celebrated theorem) are special cases of results on first order expansions of $(\mathbb{R}, <, +, 0)$. One result along these lines is that any continuous function $[0, 1] \to \mathbb{R}$ recognizable by an automaton is generically affine in a certain sense, improving several results in the literature. I anticipate this result to be a special case of a theorem on attractors of linear iterated function systems.

1.2. **Interpolative fusions.** The theory of a structure is **model complete** if every definable set may be defined by a formula that only uses existential quantifiers. This notion is of central importance to model theory. Every structure that is said to be "tame" is either model complete or near model complete in some sense. When we show that a first order structure has a positive model-theoretic property, we first show that the structure satisfies some appropriate version of model completeness and then obtain the property as a consequence.

There is no general theory of model complete structures. One can always extend the language of a first order structure $\mathfrak{M}$ to force model completeness without changing the collection of definable sets. The assumption of model completeness therefore yields no information about definable sets.

I believe it is possible to create a general theory of model completeness by studying transformations of theories which preserve model completeness, methods of constructing new model complete theories from old. Many well-known results about specific individual theories should be understood as specializations of theorems about such construction methods.

The interpolative fusion program focuses on a versatile method of constructing new model complete theories: given model complete theories T_1 and T_2 in distinct first order languages L_1 and L_2 we can (under certain conditions) construct a model complete theory $T_\cup^*$ extending $T_\cup := T_1 \cup T_2$ such that a model of $T_\cup^*$ is a *random* or *generic* fusion of a T_1-model and a T_2-model over a model of $T := T_1 \cap T_2$. We call this $T_\cup^*$ the **interpolative fusion** of T_1 and T_2 (over T). This notion generalizes many "random" constructions in model theory. A suprising number of theories of current interest may be realized as interpolative fusions of simpler theories, often in highly non-obvious ways. Results of the form "structure $\mathfrak{M}$ has property P" are often special cases of results of the form "if T_1 and T_2 satisfy P (and possibly some extra additional conditions) then $T_\cup^*$ satisfies P". The interpolative fusion construction may also be applied to produce and analyze structures that lie in an uncharted zone of the model-theoretic universe. One example is the algebraic closure of a finite field equipped with a dense multiplicatively-invariant cyclic ordering. The study of these

[1] In first order logic, one can only quantify over elements of a structure, in second order logic one can quantify over all subsets of all cartesian powers of the structure, in monadic second order logic one can only quantify over subsets of the structure.

examples will provide inspiration for the next generation of abstract model theory and yield a new stream of applications to other areas of mathematics.

2. Expansions of $(\mathbb{R}, <, +, 0)$

We think of real numbers as elements of a field, as algebraic objects. We also consider base $r \in \mathbb{N}^{\geq 2}$ expansions (or more exotic symbolic expansions) of real numbers and think of real numbers as infinite words in a finite language, perhaps from the viewpoint of the theory of combinatorics on words. Various conjectures and results assert that the algebraic structure on $\mathbb{R}$ and the "decimal expansion" structure on $\mathbb{R}$ are somehow "orthogonal" over the additive (or linear) structure on $\mathbb{R}$. Examples are the conjecture of Borel that an irrational algebraic number is normal in every base and the Hartmanis-Stearns conjecture that the r-ary digits of an irrational algebraic number cannot be computed in linear time by a Turing machine. While our work has no immediate connection to these famous conjectures, it gives another, model-theoretic, expression of the orthogonality of algebraic structure and decimal structure over linear structure in the reals. Our work, like some work on these conjectures [1], is related to automata theory.

Throughout $\mathfrak{R}$ is a first order expansion of $(\mathbb{R}, <, +, 0)$ and "definable" without modification means "$\mathfrak{R}$-definable, possibly with parameters from $\mathbb{R}$".

2.1. **Type A.** Topological dimension does not behave well with respect to general closed sets and continuous maps. It is much better behaved on semialgebraic sets and semialgebaic maps. Semialgebraic maps cannot raise topological dimension, and topological dimenesion has good additive properties on semialgebraic sets. Dimension computations are useful in semialgebraic geometry, theorems are typically proven by some kind of induction on dimension. We have shown that topological dimension on closed sets and continuous functions definable in a type A expansion behaves similarly to the semialgebraic case, allowing the use of dimension-induction arguments. We have also shown that the good properties which hold in the type A case fail away from the type A case. For example any expansion which is not type A defines a continuous surjection from a compact zero-dimensional set to $[0, 1]$. One result proven using the dimension theory (and other tools) is the following:

Theorem 2.1. *Suppose $\mathfrak{R}$ is type A and $f : [0, 1]^m \to \mathbb{R}^n$ is continuous and definable. Fix $k \geq 1$. Then there is a dense open subset of $[0, 1]^m$ on which f is C^k.*

We now describe an important corollary to Theorem 2.1. We say $\mathfrak{R}$ is of **field-type** if there is an open interval $I \subseteq \mathbb{R}$ and continuous definable $\oplus, \otimes : I^2 \to I$ and $0_I, 1_I \in I$ such that $(I, <, \oplus, \otimes, 0_I, 1_I)$ is isomorphic to $(\mathbb{R}, <, +, \cdot, 0, 1)$.

Theorem 2.2. *Suppose that $\mathfrak{R}$ is type A and not of field-type. Then every continuous definable function $[0, 1]^m \to \mathbb{R}^n$ is locally affine on a dense open subset of $[0, 1]^k$.*

We say that $\mathfrak{R}$ is **linear type** if $\mathfrak{R}$ is type A and not of field-type. One current goal is to finish writing a proof of Theorem 2.3, a first step towards a cell decomposition for type A expansions.

Theorem 2.3. *Suppose $\mathfrak{R}$ is type A. Suppose $X \subseteq \mathbb{R}^n$ is closed, definable, and of topological dimension $0 \leq d \leq n$. Fix $k \geq 1$. Then there are definable nonempty open sets $U \subseteq \mathbb{R}^n$, $V_1 \subseteq \mathbb{R}^d$, $V_2 \subseteq \mathbb{R}^{n-d}$, a definable non-empty compact zero-dimensional $E \subseteq V_2$, and a C^k homeomorphism $h : V_1 \times V_2 \to U$ which restricts to a homeomorphism between $V_1 \times E$ and $X \cap U$. If $\mathfrak{R}$ is linear type then we additionally have that the image of $V_1 \times \{p\}$ is the intersection of a hyperplane with U for all $p \in E$.*

The results of [7, 9] show Theorem 2.3 is close to optimal. Theorem 2.3 shows that if $X \subseteq \mathbb{R}^n$ is virtually any fractal of positive topological dimension then $(\mathbb{R}, <, +, 0, X)$ is not type A. We now describe a type B expansion which defines fractals such as the Menger sponge.

2.2. **Type B.** We describe an important type B expansion (see [3]). Fix $r \in \mathbb{N}^{\geq 2}$. Let V_r be a 3-ary relation on $\mathbb{R}$ where $V_r(x, u, k)$ holds when $u = r^n$ and u is the nth digit in some base r expansion of x. Let $\mathfrak{V}_r$ be $(\mathbb{R}, <, +, 0, V_r)$. The theory of $\mathfrak{V}_r$ is bi-interpretable with S1S and thus decidable. Buchi's theorem may be used to show that a subset of $\mathbb{R}^n$ is $\mathfrak{V}_r$-definable (without parameters) if and only if there is an automaton that recognizes the set of base r expansions of elements of X. All known type B expansions are associated to some enumeration system on the real numbers in an analogous fashion. I now describe some of our general results on type B expansions.

A definable set is **ω-orderable** if it admits a definable order with order-type $(\mathbb{N}, <)$ and a **dense ω-order** is a definable ω-orderable subset of $\mathbb{R}$ which is dense in some open interval. Note that an expansion is type A if it does not admit a dense ω-order. I think of the following theorem as stating that type B expansions are of "decimal expansion type". Here $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$ and Ω^D is the set of all functions $D \to \Omega$. The first order theory of the two-sorted first order structure $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, <)$ is canonically identified with S1S.

Theorem 2.4. *Suppose $\mathfrak{R}$ admits a dense ω-order. Then the structure induced on $[0, 1]$ by $\mathfrak{R}$ is bi-interpretable with a first order expansion of $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, <)$. Furthermore, there is a definable finite set Ω, definable ω-orderable set D, and a definable injection*

$$\vartheta : [0, 1] \to \Omega^D.$$

The function ϑ definably codes real numbers as infinite words in a finite language Ω. The first claim of Theorem is proven in [10], the latter two claims follow by slight extensions of the arguments of [10]. The following theorem, a restatement of a result of Hieronymi [8], expresses the orthogonality of tame expansions of field-type and type B expansions.

Theorem 2.5. *If $\mathfrak{R}$ admits a dense ω-order and is of field type then $\mathfrak{R}$ is type C. Equivalently, a type B expansion cannot be of field-type.*

A complete proof of Theorem 2.3, together with Theorem 2.5 and results from [11], would yield:

Corollary 2.6. *Suppose $X \subseteq \mathbb{R}^n$ is closed of topological dimension $0 \leq d < n$. Suppose X is definable in a type B expansion of $(\mathbb{R}, <, +, 0)$ and is also definable in some type A expansion of $(\mathbb{R}, <, +, 0)$. Then there is a subset W of X which is dense and open in X such that every $p \in W$ has an open neighbourhood $U \subseteq \mathbb{R}^n$ satisfying*

$$U \cap X = T(V \times E)$$

for some definable open $V \subseteq \mathbb{R}^d$, definable compact zero-dimensional $E \subseteq \mathbb{R}^{n-d}$, and $T \in \mathrm{Gl}_n(\mathbb{R})$.

Corollary 2.6 asserts the orthogonality of type A expansions and type B expansions over linear type expansions. It says that any closed set definable in both a type A expansion and a type B expansion is in some sense an "affine object".

2.2.1. *Two questions on type B expansions.* There are two important interrelated questions on type B expansions, one precise and one vague. We state the first in three (non trivially) equivalent ways.

Question 2.7. *Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and definable.*

(1) *If f is nowhere C^k for some $k \geq 1$, then must $\mathcal{R}$ be type C?*
(2) *If f is nowhere locally affine then must $\mathcal{R}$ be of field-type?*

(3) If $\mathcal{R}$ is type B then must f be locally affine on a dense open subset of $[0,1]$?

There is an interesting weak version of Question 2.7: If $\mathfrak{R}$ defines a continuous surjection $[0,1] \to [0,1]^2$ is $\mathfrak{R}$ necessarily type C? The best result towards of Question 2.7(3) is the following [11]:

Theorem 2.8. *Suppose $\mathcal{R}$ is type B and $f : [0,1] \to \mathbb{R}$ is continuous and definable. Then every nonempty open subinterval I of $[0,1]$ contains x, y such that for some $0 < \delta < \frac{1}{2}|x-y|$ we have*

$$f(x+\varepsilon) - f(x) = f(y+\varepsilon) - f(y) \quad \textit{for all} \quad \varepsilon < |\delta|.$$

We obtained an affirmative result over $\mathfrak{V}_r$. Theorem 2.9 was proven by Philipp and myself together with a group of undergraduates in an REU program run through the Illinois Geometry Lab in Spring 2018.

Theorem 2.9. *Every $\mathfrak{V}_r$-definable continuous function $[0,1] \to \mathbb{R}$ is locally affine on a dense open subset of $[0,1]$. It follows that every continuous function $[0,1] \to \mathbb{R}$ whose graph is recognizable by an automaton is locally affine on a dense open subset of $[0,1]$.*

This generalizes [2] [14] [17]. The proof of Theorem 2.9 follows the proof of a theorem of Hutchinson [12, Remark 3.4] on attractors of linear iterated function systems. Both results should be special cases of some more general fact about such attractors (which I would like to isolate and prove). Such a result would allow us to generalize Theorem 2.9 to cover all known examples of type B structures. In this light Question 2.7 is closely related to the following vague question:

Question 2.10. *Are there type B expansions fundamentally unlike the known examples?*

Rabin [18] showed that the monadic second order theory of an infinite binary tree (**S2S**) is decidable, this theory is known to be strictly more complex than S1S. Does S2S interpret any interesting type B expansions? Rabin (see [27]) posed a vague question which can be understood as: Does S2S interpret $(\mathbb{R}, <, +, \cdot, 0, 1)$? One might hope to address this question by studying expansions of $(\mathbb{R}, <, +, 0)$ interpretable in S2S.

2.3. **A question of Shelah and Seese.** If $\mathfrak{R}$ is type B then the structure induced on $[0,1]$ by $\mathfrak{R}$ is bi-interpretable with a (first order) expansion of $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, <)$. Thus, it is natural to consider the following question in connection with Question 2.10: What first order expansions of $(\mathbb{N}, <)$ have decidable monadic second order theories?

This relates to an informal conjecture of Shelah that if T is a complete first order theory, and the monadic second order theory of models of T does not interpret full second order logic, then models of T are "not much more complicated than trees" [20, Problem 2.8]. I do not know of any model-theoretic work on this conjecture. A related graph-theoretic conjecture has been seriously studied. Seese [19] conjectured that if the monadic second order theory of a class $\mathcal{C}$ of finite graphs is decidable then the elements of $\mathcal{C}$ have clique width $\leq k$ for some k (and are thus "not much more complicated than trees" in many respects).

Progress towards Seese's conjecture has relied on deep theorems from graph minor theory. Shelah's remarks suggests another approach. Shelah's argument shows that if the monadic second order theory of $\mathcal{C}$ is decidable then the first order theory of $\mathcal{C}$ satisfies an extremely strong form of the model-theoretic property NIP. Since the publication of [20] the technical theory of NIP theories has been extensively developed [21, 22]. Recently, model-theoretic tameness properties related to NIP have been show to imply strengthened versions of Szemeredi regularity in certain settings [15, 6, 23]. It is reasonable to hope that the strong form of NIP defined in [20] implies another strong form of Szemeredi regularity. Any such result would translate to a result on classes of finite graphs with decidable monadic second order theory. Optimistically, one hopes this line of thought will lead to a connection between the subject of NIP theories and finite model theory.

3. Interpolative Fusions

3.1. **The basic idea.** We first recall the notion of a model companion of a first order theory T in a language L. A theory T^* in L is a **model companion** of T if T^* is model complete, every T-model embeds into a T^*-model, and every T^*-model embeds into a T-model. The theory of algebraically closed fields is the model companion of the theory of fields and the theory of $(\mathbb{R}, <, +, \cdot, 0, 1)$ is the model companion of the theory of ordered fields. In general T^* is a model companion of T if and only if T^* axiomatizes the collection of T-models which are (in a sense that can be made precise) "algebraically closed in the category of T-models". The notion of a model companion isolates structures that play the role of algebraically closed fields in various subjects.

Let I be an index set, $\{L_i\}_{i \in I}$ be a family of first order languages with pairwise intersection L, and let T_i be a model complete L_i-theory for each $i \in I$. We suppose there is an L-theory $T_\cap$ such that $T_i \cap T_j = T_\cap$ when $i \neq j$. The interpolative fusion of $(T_i)_{i \in I}$ over $T_\cap$ is the model companion $T^*_\cup$ of $T_\cup := \bigcup_{i \in I} T_i$ (if such a model companion exists).

Let $\mathfrak{M}_\cup$ be a model of $T_\cup$ and $\mathfrak{M}_\square$ be the reduct of $\mathfrak{M}_\cup$ to L_i for $\square \in I \cup \{\cap\}$. Informally, $\mathfrak{M}_\cup$ is a model of $T^*_\cup$ if and only if it is a "generic" or "random" fusion of the family $(\mathfrak{M}_i)_{i \in I}$ over $\mathfrak{M}_\cap$.

3.2. **The initial motivating example.** This project began with Minh Tran's study [25] of a specific structure. I asked a question which focused his attention on this structure. Fix a prime p and let $(\mathbb{F}, +, \cdot)$ be an algebraic closure of the field with p elements. Let $(\mathbb{F}^\times, \cdot, 1)$ be the multiplicative group of $\mathbb{F}$ and $\chi : (\mathbb{F}^\times, \cdot, 1) \to (\mathbb{C}^\times, \cdot, 1)$ be an injective character. Then χ takes values in the unit circle. Let $\lhd$ be the pullback under χ of the natural cyclic order on the unit circle. Then $(\mathbb{F}^\times, \cdot, 1, \lhd)$ is a cyclically ordered abelian group (much like a linearly ordered abelian group). Minh showed that the theory of $(\mathbb{F}, \cdot, +, \lhd)$ is model complete and essentially showed this theory is the interpolative fusion of the theories of $(\mathbb{F}, \cdot, 0, 1, \lhd)$ and $(\mathbb{F}, \cdot, +, 0, 1)$ over the theory of $(\mathbb{F}, \cdot, 0)$.

Minh observed that his proof of model completeness resembled several existing model completeness proofs. I encouraged him to look for a common generalization, he found one, which we refined to produce a general set of conditions on the individual T_i which ensure the existence of $T^*_\cup$. We subsequently realized that these conditions are satisfied in many situations of interest.

3.3. **Other examples.** We give a non-exhaustive list of examples of interpolative fusions. Several "generic" constructions of interest fall under the framework of interpolative fusions. For example the theory of the random graph in the interpolative fusion of two theories, each of which is mutually interpretable with the theory of an infinite set with equality. This sheds new light on one of the most important and best understood examples in model theory. Examples include the expansion of a theory by a generic unary predicate or generic automorphism studied in [5] , or the generic skolemization of a theory studied in [26]. In the latter two cases the realization that the theory may be decomposed as an interpolative fusion is non-obvious. Alexi Block Gorman (Phd student at UIUC) has been investigating the expansion of an o-minimal structure by a generic subgroup, one specific example is the expansion of $(\mathbb{R}, <, +, \cdot, 0, 1)$ by a generic subgroup of $\mathbb{R}^\times$. This theory is essentially the interpolative fusion of the theory of $(\mathbb{R}^\times, \cdot, <, +, 0, 1)$ and the theory of $(\mathbb{R}^\times, \cdot, 0, G)$ (where G is a divisible subgroup of infinite index) over the theory of $(\mathbb{R}^\times, \cdot, 0)$.

The theory of differentially closed fields is the interpolative fusion of two theories, both of which are bi-interpretable with the theory of algebraically closed fields. This result generalizes to Moosa and Scanlon's D-rings [16].

Let P be the set of prime numbers and additive inverses of primes. Shelah and Kaplan [13] show (assuming Dickson's conjecture) that $(\mathbb{Z}, +, 0, P)$ is tame (decidable and supersimple). Minh and I hope to show that Dickson's conjecture is equivalent to the assertion that the theory of $(\mathbb{Z}, +, 0, P)$ is an interpolative fusion of two particular theories which we expect to be stable.

Given an injective character $\chi : (\mathbb{Z}, +) \to \mathbb{C}^\times$ we declare $\triangleleft_\chi$ be the pullback by χ of the cyclic order on the unit circle. Minh and I studied $(\mathbb{Z}, +, 0, \triangleleft_\chi)$ and showed that it is tame (NIP, etc) [24]. Kronecker's approximation theorem should imply that if χ, ψ are linearly independent then the theory of $(\mathbb{Z}, +, 0, \triangleleft_\chi, \triangleleft_\psi)$ is the interpolative fusion of the theories of $(\mathbb{Z}, +, 0, \triangleleft_\chi)$ and $(\mathbb{Z}, +, 0, \triangleleft_\psi)$ over the theory of $(\mathbb{Z}, +, 0)$. Such expansions define Bohr subsets of $\mathbb{Z}$ and so relate to recent unpublished work of Conant, Pillay, and Terry on NIP groups and Bohr sets.

3.4. **Preservation of neo-stability properties.** For the past several decades abstract model theory has focused on combinatorical tameness properties defined by Shelah (stability, simplicity, NIP, NSOP, NTP_2,...). These properties assert that definable sets cannot form certain combinatorial configurations (a clarifying general framework is given in [20, 5.15]). The definitions of these properties are not initially natural. They are justified by their deep consequences on the structure of definable sets. Classifying tame first order theories according to these properties gives a beautiful map of the universe of tame structures.

Many of the examples described above satisfy one or another of these properties. In each case this follows (or should follow) from a general theorem of the following form: if each T_i has P (and other conditions are satisfied) then $T^*_\cup$ has P. One result in this direction is the following Theorem. The idea is due to Minh and myself and the proof is due to Alex.

Theorem 3.1. *Suppose that each T_i is NSOP_1 and that $T_\cap$ is stable, 3-unique*[2]*, and has weak elimination of imaginaries. Then $T^*_\cup$ is NSOP_1.*

Theorem 3.1 is an important contribution to the emerging subject of NSOP_1 theories. It gives a strong realization of the idea that NSOP_1 is be preserved under generic constructions. We also give (more complicated) necessary and sufficient conditions for $T^*_\cup$ to be simple. This yields a mutual generalization of the facts that the expansion of a stable theory by a generic predicate and the expansion of a stable theory by a generic automorphism are simple (when they exist) [5]. One current goal is to complete a preservation proof for NTP_2.

3.5. **The frontier.** The machinery of interpolative fusions allows us to take some steps into an unexplored region of the model-theoretic universe: structures that are SOP and TP_2 (and thus do not satisfy any studied combinatorial tameness property) but which are in some sense "NIP + random". Interpolative fusions of unstable NIP theories are typically SOP and TP_2. Examples include generic skolemizations of unstable NIP theories, the expansion of $(\mathbb{R}, <, +, \cdot, 0, 1)$ by a generic subgroup of $\mathbb{R}^\times$, and $(\mathbb{F}, \cdot, +, 0, 1, \triangleleft)$.

A study of the examples above will motivate future work on combinatorial tameness conditions satisfied by "NIP + random" theories. We have isolated several weakenings of NTP_2 satisfied by these examples. A test question for these properties is if they are satisfied by Keisler randomizations of unstable NIP theories.

3.6. **A forgotten definition of Shelah.** Shelah defined a property, "straight maximality" whose negation is a natural candidate for a minimal combinatorial tameness principle [20, Definition 5.20]. If a theory is straight maximal then there is a formula that defines (in sense that can be

[2]3-uniquness is a stability theoretic assumption satisfied by most familiar stable theories.

made precise) any combinatorical configuration. Shelah is sure this notion gives an interesting dividing line and says it should be possible to prove many things about straight maximality, but also says that he does not know of a natural test problem. To my knowledge no work has been done on this concept in the nineteen years since it was introduced.

The following test question is natural: If each T_i is not straight maximal then must $T^*_\cup$ also not be straight maximal? Our incomplete work towards an NTP_2 preservation result suggests this question is tractable.

References

1. Boris Adamczewski, Yann Bugeaud, and Florian Luca, *Sur la complexité des nombres algébriques*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 1, 11–14. MR 2075225
2. V. S. Anashin, *Smooth finitely computable functions are affine, or why quantum systems are wave systems*, Dokl. Akad. Nauk **465** (2015), no. 1, 11–13. MR 3496254
3. Bernard Boigelot, Stéphane Rassart, and Pierre Wolper, *On the expressiveness of real and integer arithmetic automata (extended abstract)*, Proceedings of the 25th International Colloquium on Automata, Languages and Programming (London, UK, UK), ICALP '98, Springer-Verlag, 1998, pp. 152–163.
4. J. Richard Büchi, *On a decision method in restricted second order arithmetic*, Logic, Methodology and Philosophy of Science (Proc. 1960 Internat. Congr .), Stanford Univ. Press, Stanford, Calif., 1962, pp. 1–11. MR 0183636 (32 #1116)
5. Z. Chatzidakis and A. Pillay, *Generic structures and simple theories*, Ann. Pure Appl. Logic **95** (1998), no. 1-3, 71–92. MR 1650667
6. Artem Chernikov and Sergei Starchenko, *Regularity lemma for distal structures*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 10, 2437–2466. MR 3852184
7. Harvey Friedman, Krzysztof Kurdyka, Chris Miller, and Patrick Speissegger, *Expansions of the real field by open sets: definability versus interpretability*, J. Symbolic Logic **75** (2010), no. 4, 1311–1325. MR 2767970 (2012a:03100)
8. Philipp Hieronymi, *Defining the set of integers in expansions of the real field by a closed discrete set*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2163–2168. MR 2596055 (2011j:03086)
9. ———, *A tame cantor set*, J. European Math. Soc. to appear (2016).
10. Philipp Hieronymi and Erik Walsberg, *Interpreting the monadic second order theory of one successor in expansions of the real line*, Israel J. Math. **to appear, arXiv:1601.04555** (2016).
11. ———, *On continuous functions definable in expansions of the ordered real additive group*, **arXiv:1709.03150** (2017).
12. John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747. MR 625600
13. Itay Kaplan and Saharon Shelah, *Decidability and classificaton of the theory of integers with primes*, J. Symb. Log. **82** (2017), no. 3, 1041–1050. MR 3694340
14. Michal Konečný, *Real functions computable by finite automata using affine representations*, Theoret. Comput. Sci. **284** (2002), no. 2, 373–396, Computability and complexity in analysis (Castle Dagstuhl, 1999). MR 1923906
15. M. Malliaris and S. Shelah, *Regularity lemmas for stable graphs*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1551–1585. MR 3145742
16. Rahim Moosa and Thomas Scanlon, *Model theory of fields with free operators in characteristic zero*, J. Math. Log. **14** (2014), no. 2, 1450009, 43. MR 3304121
17. Jean-Michel Muller, *Some characterizations of functions computable in on-line arithmetic*, IEEE Trans. Comput. **43** (1994), no. 6, 752–755. MR 1284161
18. Michael O. Rabin, *Decidability of second-order theories and automata on infinite trees.*, Trans. Amer. Math. Soc. **141** (1969), 1–35. MR 0246760 (40 #30)
19. D. Seese, *The structure of the models of decidable monadic theories of graphs*, Ann. Pure Appl. Logic **53** (1991), no. 2, 169–195. MR 1114848
20. Saharon Shelah, *On what I do not understand (and have something to say), model theory*, Math. Japon. **51** (2000), no. 2, 329–377. MR 1747306
21. Pierre Simon, *A guide to nip theories*, Lecture Notes in Logic, vol. 44, Cambridge University Press, 2015.
22. ———, *Linear orders in nip structures*, **arXiv:1807.07949** (2018).

23. Terence Tao, *Expanding polynomials over finite fields of large characteristic, and a regularity lemma for definable sets*, Contrib. Discrete Math. **10** (2015), no. 1, 22–98. MR 3386249
24. Minh Tran and Erik Walsberg, *A family of dp-minimal expansions of* $(\mathbb{Z};+)$, **arxiv.org/abs/1711.04390** (2017).
25. Minh Chieu Tran, *Tame structures via multiplicative character sums on varieties over finite fields*, **arXiv:1704.03853** (2017).
26. Peter M. Winkler, *Model-completeness and Skolem expansions*, (1975), 408–463. Lecture Notes in Math., Vol. 498. MR 0540029
27. Faried Abu Zaid, Erich Grädel, Ł ukasz Kaiser, and Wied Pakusa, *Model-theoretic properties of* ω*-automatic structures*, Theory Comput. Syst. **55** (2014), no. 4, 856–880. MR 3259503