SOME QUESTIONS

It is possible that some of these questions already have answers.

Question 0.1. *Does ZFC imply that there is a subring of* $(\mathbb{R}, +, \cdot, 0, 1)$ *with Hausdorff dimension greater than zero and less than one?*

Background: Erdos and Volkmann [7] showed that for every $0 < \alpha < 1$ there is a Borel subgroup of $(\mathbb{R}, +, 0)$ with Hausdorff dimension α. They raised the question of whether there are subrings of intermediate dimension. Edgar-Miller [6] showed that there are no analytic subrings of intermediate dimension. Around the same time Bourgain gave an exponentially more difficult proof that there are no Borel subrings of intermediate dimension [4]. The Edgar-Miller proof in fact shows that ZF+AD implies that *every* subring has Hausdorff dimension zero or one. Maudlin [11] shows that ZFC+CH implies that for every $0 < \alpha < 1$ there is a subring of dimension α. If Question 0.1 has a positive answer then subrings of intermediate dimension are "choice pathologies" in a precise sense. If Question 0.1 has a negative answer then the existence of subrings of intermediate dimension should be tied to cardinal invariants of the continuum or to forcing axioms.

Question 0.2. *Is there a Borel subring of* $(\mathbb{R}, +, \cdot, 0, 1)$ *with packing dimension* > 0 *and* < 1*?*

Background: This seems to be a very natural question in light of the work on Hausdorff dimensions of Borel subrings discussed above. I am not aware of any work whatsoever on this topic.

Question 0.3. *Can an NIP expansion of* $(\mathbb{N}, <)$ *interpret an infinite field?*

Background: A result of Simon and Shelah [14] easily implies that an NIP expansion of $(\mathbb{N}, <)$ cannot interpret an infinite-dimensional vector space over a finite field. It follows that an NIP expansion of $(\mathbb{N}, <)$ cannot interpret an infinite field of positive characteristic. I find this question interesting for several reasons. One is that everyone seems to agree that it should have a positive answer, but no one can make any progress whatsoever. A solution to this problem might only require a single clever idea, or it might involve a substantial deepening of our understanding of NIP theories.

Question 0.4. *Fix* $0 < \alpha < 1$*. For each* $n \geq 1$ *we create a subset* P_n *of* $\mathbb{Z}/n\mathbb{Z}$ *by independently picking elements with probability* $n^{-\alpha}$*. Does* $\mathcal{Z}_n = (\mathbb{Z}, +, 0, P_n)$ *satisfy a zero-law? That is, for every formula* ϕ *do we necessarily have*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n \models \phi) = 0 \quad \textit{or} \quad \lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n \models \phi) = 1 \quad ?$$

Background: This question is motivated by the remarkable work of Shelah and Spencer [15] on zero-one laws for $G(n, n^{-\alpha})$. This works suggests that one might have to be careful about the choice of α. This work is also connected to Hrushovski constructions [2] [8]. So one might even go so far as to hope that if this problem admits a positive solution then one might be able to use some kind of Hrushovksi construction of construct models of the limit theory. This question is inspired by some comments of Anton Bernshteyn.

Question 0.5. *Suppose $(R, <, +, \cdot, 0, 1)$ is an ordered field. If $(R, <, +, \cdot, 0, 1)$ does not admit a definable convex valuation then must $(R, <, +, \cdot, 0, 1)$ be elementarily equivalent to an archimedean ordered field?*

Background: It is easy to see that an ordered field defines a convex additive subgroup iff it defines a convex subring (and any convex subring is a valuation ring of a convex valuation). It follows from work of Robinson and Zakon [13] that an ordered abelian group is elementarily equivalent to an archimedean ordered abelian group iff it does not admit a convex definable subgroup. Question 0.5 holds for Henselian fields: a Henselian ordered field does not admit a definable convex valuation iff it is real closed (Henselian archimedean ordered fields are real closed).

Question 0.6. *Is there a model-theoretically tame infinite field which is not large?*

Background: See Pop [12] for a nice survey on large fields. I do not know of a good candidate for a model-theoretically tame field which is not large. Let me know if you have one.

Question 0.7. *To what extent can known results in transcendental number theory be used to decide the theory of $(\mathbb{R}, <, +, \cdot, \exp, 0, 1)$?*

Background: Once upon a time Tarski asked if the theory of $(\mathbb{R}, <, +, \cdot, \exp, 0, 1)$ is decidable. Macintrye and Wilkie [10] famously showed that Schanuel's conjecture implies that this theory is decidable, and indeed decidability is equivalent to a weak form of Schanuel's conjectures. Thus decidability of $(\mathbb{R}, <, +, \cdot, \exp, 0, 1)$ is out of reach. Almagor, Christikov, Ouaknine, and Worrell [1] in effect use a strong result of transcendental number theory (Baker's theorem) to decide a fragment of the theory of $(\mathbb{R}, <, +, \cdot, \exp, 0, 1)$. It would be nice to know exactly how much of this theory Baker's theorem decides, and if there are other results in transcendental number theory that decide more of the theory.

Question 0.8. *For which o-minimal expansions $\mathcal{R}$ of $(\mathbb{R}, <, +)$ is $(\mathcal{R}, \mathbb{Q})$ model theoretically tame?*

Background: If we replace $\mathbb{Q}$ with $\mathbb{Z}$ than the question becomes much easier and I already know the answer. It is well known that $(\mathbb{R}, <, +, \cdot, \mathbb{Q})$ defines the set of integers and is hence wild. It is also well known that $(\mathbb{R}, <, +, \mathbb{Q})$ admits quantifier elimination (in a suitable language) and is thus tame. It was recently shown by Block Gorman that $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, \mathbb{Q})$ is tame. We let $t^{\mathbb{Q}} = \{t^q : q \in \mathbb{Q}\}$ for any positive real number t. It was shown in [5] that $(\mathbb{R}, <, +, \cdot, t^{\mathbb{Q}})$ is tame for any t. Note that $\log_t$ gives an isomorphism

$$(\mathbb{R}^{>0}, <, \cdot, 1, t, t^{\mathbb{Q}}) \to (\mathbb{R}, <, +, 0, 1, \mathbb{Q}).$$

Therefore $\log_t$ gives an isomorphism between $(\mathbb{R}^{>0}, <, \cdot, +, t^{\mathbb{Q}})$ and a structure of the form $(\mathcal{R}, \mathbb{Q})$ where $\mathcal{R}$ is some o-minimal expansion of $(\mathbb{R}, <, +)$. Note that the question "For which o-minimal expansions $\mathcal{R}$ of $(\mathbb{R}, <, +, \cdot)$ is $(\mathcal{R}, t^{\mathbb{Q}})$ tame" is a special case of 0.8. If $\mathcal{R}$ satisfies a "Mordell-Lang" property with respect to finitely rank subgroups of $(\mathbb{R}^{>0}, \cdot)$ (as $(\mathbb{R}, <, +, \cdot)$ does) than $(\mathcal{R}, t^{\mathbb{Q}})$ should be tame. It is known that $(\mathbb{R}_{\text{an}}, t^{\mathbb{Q}})$ is wild (defines the integers). I believe it is likely that $(\mathcal{R}, t^{\mathbb{Q}})$ is tame when $\mathcal{R}$ is the expansion of $(\mathbb{R}, <, +, \cdot)$ by a generic smooth function $[0, 1] \to \mathbb{R}$ studied in [9].

Suppose $\mathcal{Q}$ is a weakly o-minimal expansion of $(\mathbb{Q}, <, +)$. Let $\mathcal{R}$ be the expansion of $(\mathbb{R}, <, +)$ by an n-ary predicate defining the closure in $\mathbb{R}^n$ of each $\mathcal{Q}$-definable subset of $\mathbb{Q}^n$. An adaptation of [3] shows that $\mathcal{R}$ is o-minimal, $(\mathcal{R}, \mathbb{Q})$ is tame, and the structure on $\mathbb{Q}$ induced by $(\mathcal{R}, \mathbb{Q})$ is a reduct of the Shelah expansion of $\mathcal{Q}$ (in particular is weakly o-minimal). Thus 0.8 is closely tied to the question "What are the weakly o-minimal expansions of $(\mathbb{Q}, <, +)$". It is possible to show that an expansion of $(\mathbb{Q}, <, +)$ is weakly o-minimal if and only if it is dp-minimal, so we may replace "weakly o-minimal" with "dp-minimal". This is nice because dp-minimality is a more general and abstract condition.

We can also generalize the question by replacing $\mathbb{Q}$ with a different finite rank divisible additive subgroup of $\mathbb{R}$.

Question 0.9 should be closely related to Question 0.8:

Question 0.9. *For which real analytic $f : [0, 1] \to \mathbb{R}$ is $(\mathbb{R}, <, +, \cdot, 0, 1, f(\mathbb{Q}))$ model theoretically tame?*

Background: One can also replace $\mathbb{Q}$ with a general number field. Note that if f is Nash then $(\mathbb{R}, <, +, \cdot, 0, 1, f(\mathbb{Q}))$ trivially defines $\mathbb{Q}$. If f is the restriction of an exponential $(\mathbb{R}, <, +, \cdot, 0, 1, f(\mathbb{Q}))$ is tame.

Question 0.10. *What groups are interpretable in the theory of boolean algebras?*

Background: This question seems very natural.

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