THE MARKER-STEINHORN THEOREM VIA DEFINABLE LINEAR ORDERS

Abstract. We give a short proof of the Marker-Steinhorn theorem for o-minimal expansions of ordered groups. The key tool is Ramakrishnan’s classification of definable linear orders in such structures.

1. Introduction

Let \( M = (M, \leq, \ldots) \) be an o-minimal expansion of a dense linear order without endpoints, possibly with additional structure, in the language \( \mathcal{L} \). A type \( p(x) \) over \( M \) is definable if for every \( \mathcal{L} \)-formula \( \delta = \delta(x, y) \) in the (object) variables \( x = (x_1, \ldots, x_n) \) and (parameter) variables \( y = (y_1, \ldots, y_n) \) there is a defining formula for the restriction \( p \upharpoonright \delta \), i.e., a formula \( \phi(y) \), possibly with parameters from \( M \), such that \( \delta(x, b) \in p \iff M \models \phi(b) \), for all \( b \in M^n \).

A set \( C \subseteq M \) is a cut in \( M \) if whenever \( c \in C \), then \((-\infty, c] := \{ a \in M : a < c \} \) is contained in \( C \). Let \( \delta(x, y) \) be the formula \( x > y \) (in the language of \( M \)). It is well-known that cuts in \( M \) correspond in a one-to-one way to complete \( \delta \)-types over \( M \), where to the cut \( C \) in \( M \) we associate the complete \( \delta \)-type \( p_C(x) := \{ \delta(x, b) : b \in C \} \cup \{ \neg \delta(x, b) : b \in M \setminus C \} \).

The \( \delta \)-type \( p_C \) is definable if and only if the cut \( C \) in \( M \) is definable. If \( C \) is of the form \((-\infty, c] := \{ a \in M : a \leq c \} \) (for some \( c \in M \)) or \((-\infty, c)c \in M \) or \( \{ \pm \infty \} \) then \( C \) is definable. Such cuts are said to be rational. It follows from o-minimality that all definable cuts are rational. If \((M, \leq) = (\mathbb{R}, \leq) \) then all cuts in \( M \) are rational. This can be used to define the standard part map for elementary extensions. That is, if \((M, \leq) = (\mathbb{R}, \leq) \) and \( M \leq M^* = (M^*, \leq, \ldots) \) then we define the standard part map

\[
b \mapsto \sup\{ a \in M : a \leq b \} : M^* \cup \{ \pm \infty \} \to M \cup \{ \pm \infty \},
\]

where we declare \( \sup \emptyset := -\infty \) and \( \sup M := +\infty \). To generalize this, we say an elementary extension \( M \leq M^* \) is tame over \( M \) if for every \( a \in M^* \) the cut \( \{ b \in M : b \leq a \} \) is rational. (Thus if \((M, \leq) = (\mathbb{R}, \leq) \) then every elementary extension of \( M \) is tame over \( M \).) We then define the standard part map in the same way.

It follows by o-minimality that every 1-type over \( M \) is determined by its restriction to \( \delta \), so a 1-type over \( M \) is definable exactly when the associated cut in \( M \) is rational. It trivially follows that \( M \leq M^* \) is tame over \( M \) if and only if for every \( a \in M^* \), the type \( tp(a|M) \) is definable. Marker and Steinhorn [3] generalized this to show that if \( M \leq M^* \) is tame over \( M \) then for every \( a \in (M^*)^m \), the type \( tp(a|M) \) is definable. In particular if \((M, \leq) = (\mathbb{R}, \leq) \) then every type over \( \mathbb{R} \) is definable. See [9] for geometric applications of this useful result. The original proof of Marker and Steinhorn uses a complicated inductive argument. Tressl [7] proved the Marker-Steinhorn for o-minimal expansions of real closed fields with a short
and clever argument using valuation theory and co-heirs that gives little idea as to
the form of the defining formula of a type. Chernikov and Simon gave a proof using
NIP-theoretic machinery [2]. We give a more constructive proof of the Marker-
Steinhorn Theorem for o-minimal expansions of ordered groups. The crucial idea
is to reduce the analysis of $n$-types to an analysis of cuts in definable linear orders.
Our main tool is the following theorem of Ramakrishnan [5], which is closely related
to earlier work of Onshuus-Steinhorn [4]. Let $\leq_{\text{lex}}$ be the lexicographic order on $M^k$.

**Theorem 1.1.** Suppose $\mathcal{M}$ expands an ordered group. Then every definable linear
order is definably isomorphic to a definable subset of some $M^k$ equipped with the
induced lexicographic order.

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version of the proof.

2. Conventions

Throughout, $\mathcal{M}$ is an o-minimal expansion of an ordered abelian group, and
$\mathcal{M} \leq \mathcal{M}^* = (M^*, \ldots)$ is tame over $\mathcal{M}$. Unless said otherwise, “definable” means
“definable, possibly with parameters,” and the adjective “definable” applied to
subsets of $M^m$ or maps $A \rightarrow M^n$, $A \subseteq M^m$, will mean “definable in $\mathcal{M}$.” The
basic facts about o-minimal structures that we use may be found in [8]. We let
$m,n,k,l$ range over natural numbers. Given sets $A,B,C \subseteq A \times B$, and $a \in A$ we
let
$$C_a := \{b \in B : (a,b) \in C\}.$$ 
If $A \subseteq M^m$ is a definable set, then $A^*$ denotes the subset of $(M^*)^m$ defined in $\mathcal{M}^*$
by the same formula (since $\mathcal{M} \leq \mathcal{M}^*$, this does not depend on the choice of a defining
formula). Similarly, if $f : A \rightarrow M^n$ is a definable map, then $f^* : A^* \rightarrow (M^*)^n$
denotes the map whose graph is defined in $\mathcal{M}^*$ by the same formula as the graph
of $f$. If $A \subseteq M^m$ is definable then $\dim(A)$ is the usual o-minimal dimension of $A$.
Given a bounded definable $A \subseteq M^*$ we let $\mu(A)$ be the sum of the lengths
of the components of $A$. We call $\mu(A)$ the measure of $A$. (Indeed, $\mu$ is a finitely
additive measure on the collection of bounded $\mathcal{M}^*$-definable subsets of $M^*$). If $A \subseteq (M^*)^m \times M^*$ is such that every $A_x$ is bounded then there is a definable
$f : (M^*)^m \rightarrow M^*$ such that $f(x) = \mu(A_x)$.

3. Cuts in Definable Linear Orders

Throughout this section $(P, \leq_P)$ is a definable linear order and $P \subseteq M^m$.

**Proposition 3.1.** If $V \subseteq P^*$ is $\mathcal{M}^*$-definable and $W = V \cap P$ is a cut in $P$,
then $W$ is definable.

The proof of this proposition is the most difficult part of this paper. The difficulty
largely lies in the fact that $V$ may not be a cut in $P^*$. We need the following three
lemmas for Proposition 3.1. The first is an easy base case of the Marker-Steinhorn
theorem, which we leave to the reader.

**Lemma 3.2.** If $A \subseteq M^*$ is $\mathcal{M}^*$-definable then $A \cap M$ is definable.

The second lemma follows easily from o-minimality, we leave the proof to the reader.
Lemma 3.3. Let $0 < q < 1$ be a rational number. Suppose $I \subseteq M$ is a bounded interval and $J \subseteq I^*$ is $\mathcal{M}^*$-definable. If $\mu(J) \geq q \mu(I^*)$ then $J \cap I$ is nonempty. If $I \subseteq J$ then $\mu(J) \geq q \mu(I^*)$.

Lemma 3.4. Suppose $A \subseteq M^m$ is definable and $B$ is an $\mathcal{M}^*$-definable subset of $A^*$ such that $A_x$ is either contained in or disjoint from $B_x$ for all $x \in M^{m-1}$. Then there is an $\mathcal{M}^*$-definable $D \subseteq (\mathcal{M}^*)^{m-1}$ such that

$$D \cap M^{m-1} = \{ x \in M^{m-1} : A_x \subseteq B_x \}.$$ 

Proof. Let $\{C_1, \ldots, C_n\}$ be a cell decomposition of $A$. Note that $(C_i)_x$ is either contained in or disjoint from $B_x$ for all $x \in M^{m-1}$ and $1 \leq i \leq n$. If $D_i$ is an $\mathcal{M}^*$-definable subset of $(\mathcal{M}^*)^{m-1}$ such that

$$D_i \cap M^{m-1} = \{ x \in M^{m-1} : (C_i)_x \subseteq B_x \} \text{ for } 1 \leq i \leq n,$$

then $D := D_1 \cap \ldots \cap D_n$ satisfies the conditions of the lemma. We therefore assume $A$ is a cell. We now consider four cases. The first case is when $A$ is the graph of a continuous definable $f : A' \to M$ on a cell $A' \subseteq M^{m-1}$. In this case we take $D$ to be the set of $x \in (A')^*$ such that $f(x) \in B_x$. The second case is when

$$A = \{ (x, t) \in M^{m-1} \times M : x \in A', f(x) < t < g(x) \}$$

for continuous definable $f, g : A' \to M$ on a cell $A' \subseteq M^{m-1}$ such that $f(x) < g(x)$ for all $x \in A'$. It follows from Lemma 3.3 that $A_x \subseteq B_x$ implies

$$\mu(B_x) \geq \frac{1}{2} \mu(A_x^*) = \frac{1}{2} [g^*(x) - f^*(x)] \text{ for all } x \in A'.$$

Lemma 3.3 also shows that for all $x \in A'$, if $\mu(B_x) \geq \frac{1}{2} \mu(A_x^*)$ then $A_x$ and $B_x$ intersect. We therefore take $D$ to be the set of $x \in (A')^*$ such that $\mu(B_x) \geq \frac{1}{2} \mu(A_x^*)$.

The third case is when

$$A = \{ (x, y) \in M^{m-1} \times M : x \in A', y > f(x) \}$$

for continuous definable $f : A' \to M$ on a cell $A' \subseteq M^{m-1}$. If $B_x$ contains $A_x$ then $B_x$ contains $\{ y \in M : f(x) < y < f(x) + 1 \}$. Conversely if $B_x$ contains $\{ y \in M : f(x) < y < f(x) + 1 \}$ then $B_x$ intersects $A_x$ and hence contains $A_x$ by assumption. Thus $B_x$ contains $A_x$ if and only if it contains $\{ y \in M : f(x) < y < f(x) + 1 \}$. Reasoning as before we take $D$ to be the set of $x \in (A')^*$ such that

$$\mu \left( A_x \cap \{ y \in M^* : f(x) < y < f(x) + 1 \} \right) \geq \frac{1}{2}.$$  

The fourth case is when

$$A = \{ (x, y) \in M^{m-1} \times M : x \in A', y < g(x) \}$$

for continuous definable $g : A' \to M$ on a cell $A' \subseteq M^{m-1}$. This case follows in the same way as the third case. \hfill \Box

We now prove Proposition 3.1.

Proof. Applying Theorem 1.1 let $P' \subseteq M^k$ be definable, $\leq_{lex}$ be the restriction of the lexicographic order on $M^k$ to $P'$, and suppose $\iota : (P, \leq_P) \to (P', \leq_{lex})$ is a definable isomorphism of linear orders. It suffices to show $\iota(W) = \iota^*(V) \cap P'$ is definable. We therefore suppose $\leq_P$ is the restriction of the lexicographic order on $M^m$ to $P$. We apply induction on $m$. If $m = 1$ then $W$ is definable by Lemma 3.2. Suppose $m \geq 2$, let $\pi : P \to M^{m-1}$ be the projection onto the
first $m - 1$ coordinates, and let $Q = \pi(P)$. Note $\pi$ is a monotone map $(P, \leq_P) \to (Q, \leq_{\text{lex}})$, it follows that $\pi(W)$ is a cut in $(Q, \leq_{\text{lex}})$. We consider two cases:

1. $\pi(W)$ has a maximum $q$ in $(Q, \leq_{\text{lex}})$,
2. $\pi(W)$ does not have a maximum in $(Q, \leq_{\text{lex}})$.

We first treat case (1). The assumption implies that $\pi^{-1}(q) \cap W$ is upwards cofinal in $W$, so $W$ is the downwards closure of $\pi^{-1}(q) \cap W$. Lemma 3.2 shows $\pi^{-1}(q) \cap W$ is definable, so $W$ is definable.

We now treat case (2). If $p \in W$ then as $\pi(p)$ is not the maximal element of $\pi(W)$ it follows that $\pi(p') >_{\text{lex}} \pi(p)$ for some $p' \in W$, which implies $p' >_p q$ for all $q \in \pi^{-1}(p)$, as $W$ is downwards closed we have $\pi^{-1}(p) \subseteq W$. Thus, for any $p \in Q$, if $W$ intersects $\pi^{-1}(p)$ then $W$ contains $\pi^{-1}(p)$. Note in particular that this implies $W = \pi^{-1}(\pi(W))$, so it suffices to show $\pi(W)$ is definable. Applying Lemma 3.4 we obtain an $\mathcal{M}^*$-definable $D \subseteq M^{m-1}$ such that $D \cap Q = \pi(W)$. As $\pi(W)$ is a cut in $(Q, \leq_{\text{lex}})$ the inductive hypothesis implies $\pi(W)$ is definable.

The proof of Proposition 3.1 may be simplified by applying a result of Shelah, see [6] or [1]. This result, which holds for any NIP structure, implies that if $D \subseteq (\mathcal{M}^*)^k$ is $\mathcal{M}^*$-definable and $\pi : (\mathcal{M}^*)^k \to (\mathcal{M}^*)^{k'}$ is a coordinate projection then there is an $\mathcal{M}^*$-definable $E \subseteq (\mathcal{M}^*)^{k'}$ such that $\pi(D \cap M^k) = E \cap M^{k'}$. Applying this result allows us to avoid the use of Lemma 3.4 and directly apply the inductive assumption to $\pi(W)$.

4. Proof of Marker-Steinhorn

Fix $b = (b_1, \ldots, b_k) \in (\mathcal{M}^*)^k$. The following theorem shows $\text{tp}(b|M)$ is definable.

**Theorem 4.1.** If $A \subseteq M^l \times M^k$ is definable then $\{a \in M^l : (a, b) \in A^*\}$ is also definable.

**Proof.** We apply induction on $k$. The base case $k = 1$ holds as all 1-types over $M$ realized in $M^*$ are definable. Suppose $k \geq 2$ and let $b' = (b_1, \ldots, b_{k-1})$. We declare $\dim(b|M)$ to be the minimal dimension of a definable $B \subseteq M^k$ such that $b \in B^*$. We first consider the case $\dim(b|M) < k$. Let $B \subseteq M^k$ be definable such that $b \in B^*$ and $\dim B < k$. Let $\{C_1, \ldots, C_n\}$ be a cell decomposition of $B$, let $1 \leq i \leq n$ be such that $b \in (C_i)^*$. After replacing $B$ with $C_i$ if necessary we suppose $B$ is a cell. As $\dim B < k$ we suppose, after permuting coordinates if necessary, that

$$B = \{(a, t) \in M^{k-1} \times M : a \in B', t = f(a)\}$$

for a cell $B' \subseteq M^{k-1}$ and a continuous definable $f : B' \to M$. Note $b_k = f^*(b')$.

Let $E$ be the set of $(a, c) \in M^l \times M^{k-1}$ such that $(a, c, f(c)) \in A$. Given $a \in M^l$, we have $(a, b) \in A^*$ if and only if $(a, b') \in E^*$. Applying the inductive hypothesis to $b'$ shows $\{a \in M^l : (a, b') \in E^*\}$ is definable. We therefore suppose $\dim(b|M) = k$.

Suppose $\{C_1, \ldots, C_n\}$ is a cell decomposition of $A$. It suffices to show that $\{a \in M^l : (a, b) \in (C_i)^*\}$ is definable for $1 \leq i \leq n$. We therefore suppose $A$ is a cell. We suppose without loss of generality that $\{a \in M^l : (a, b) \in A^*\}$ is nonempty. As $\dim(b|M) = k$ it follows that $\dim(A_x) = k$ for some $x \in M^l$. As $A$ is a cell it follows that $\dim(A_x) = k$ for all $x \in M^l$ such that $A_x \neq \emptyset$, so each $A_x$ is an open cell. Then one of the following holds:

- $A = \{(a, c, t) \in M^l \times M^{k-1} \times M : (a, c) \in A', f(a, c) < t < g(a, c)\}$,
- $A = \{(a, c, t) \in M^l \times M^{k-1} \times M : (a, c) \in A', t < g(a, c)\}$,
for a cell $A' \subseteq M^l \times M^{k-1}$ and continuous definable $f, g : A' \to M$. We only treat the third case as the previous two may be handled in the same way. In this case $(a, b) \in A^*$ if and only if $(a, b') \in (A^*)^*$ and $f^*(a, b') < b_k$. Let $D$ be the set of $a \in M^l$ such that $(a, b') \in (A^*)^*$. An application of the inductive hypothesis shows $D$ is definable. Let $\sim$ be the equivalence relation on $D$ given by $e \sim d$ if and only if $f^*(e, b') = f^*(d, b')$. The inductive hypothesis shows $\sim$ is definable. Applying the elimination of imaginaries for o-minimal expansions of ordered groups we suppose $D/\sim$ is a definable set $P$ and let $\rho : D \to P$ be the quotient map. We put a relation $\leq$ on $D$ by declaring $e \leq d$ if and only if $f^*(e, b') \leq f^*(d, b')$. The inductive hypothesis shows $\leq$ is definable. It is easy to see that $\leq$ is a quasi-order on $D$ which pushes forward to a definable linear order on $P$ under $\rho$. Abusing notation we let $\leq$ be the push-forward of $\leq$ to $P$. Let $W$ be the set of $d \in P$ for which there is an $e \in D$ such that $\rho(e) = d$ and $f^*(e, b') < b_k$. Then

$$\{ a \in M^l : (a, b) \in A^* \} = \{ a \in M^l : [a \in D] \land [\rho(a) \in W] \}. $$

It is easy to see that $W$ is a cut in $(P, \leq)$. It follows by Proposition 3.1 that $W$ is definable. □

References