THE MARKER-STEINHORN THEOREM VIA DEFINABLE LINEAR ORDERS

ABSTRACT. We give a short proof of the Marker-Steinhorn theorem for o-minimal expansions of ordered groups. The key tool is Ramakrishnan's classification of definable linear orders in such structures.

1. INTRODUCTION

Let $\mathcal{M} = (M, \leq, \dots)$ be an o-minimal expansion of a dense linear order without endpoints, possibly with additional structure, in the language $\mathcal{L}$. A type $p(x)$ over M is *definable* if for every $\mathcal{L}$-formula $\delta = \delta(x, y)$ in the (object) variables $x = (x_1, \dots, x_m)$ and (parameter) variables $y = (y_1, \dots, y_n)$ there is a defining formula for the restriction $p \restriction \delta$, i.e. a formula $\phi(y)$, possibly with parameters from M, such that $\delta(x, b) \in p \Leftrightarrow \mathcal{M} \models \phi(b)$, for all $b \in M^n$.

A set $C \subseteq M$ is a *cut* in $\mathcal{M}$ if whenever $c \in C$, then $(-\infty, c) := \{a \in M : a < c\}$ is contained in C. Let $\delta(x, y)$ be the formula $x > y$ (in the language of $\mathcal{M}$). It is well-known that cuts in $\mathcal{M}$ correspond in a one-to-one way to complete δ-types over M, where to the cut C in $\mathcal{M}$ we associate the complete δ-type

$$p_C(x) := \{\delta(x, b) : b \in C\} \cup \{\neg\delta(x, b) : b \in M \setminus C\}.$$

The δ-type p_C is definable if and only if the cut C in $\mathcal{M}$ is definable. If C is of the form $(-\infty, c] := \{a \in M : a \leq c\}$ $(c \in M)$ or $(-\infty, c)(c \in M \cup \{\pm\infty\})$, then C is definable. Such cuts are said to be *rational*. It follows from o-minimality that all definable cuts are rational. If $(M, \leq) = (\mathbb{R}, \leq)$ then all cuts in $\mathcal{M}$ are rational. This can be used to define the standard part map for elementary extensions. That is, if $(M, \leq) = (\mathbb{R}, \leq)$ and $\mathcal{M} \preceq \mathcal{M}^* = (M^*, \leq, \dots)$ then we define the standard part map

$$b \mapsto \sup\{a \in M : a \leq b\} : M^* \cup \{\pm\infty\} \to M \cup \{\pm\infty\},$$

where we declare $\sup \emptyset := -\infty$ and $\sup M := +\infty$. To generalize this, we say an elementary extension $\mathcal{M} \preceq \mathcal{M}^*$ is *tame over* $\mathcal{M}$ if for every $a \in M^*$ the cut $\{b \in M : b \leq a\}$ is rational. (Thus if $(M, \leq) = (\mathbb{R}, \leq)$ then every elementary extension of $\mathcal{M}$ is tame over $\mathcal{M}$.) We then define the standard part map in the same way.

It follows by o-minimality that every 1-type over M is determined by its restriction to δ, so a 1-type over M is definable exactly when the associated cut in $\mathcal{M}$ is rational. It trivially follows that $\mathcal{M} \preceq \mathcal{M}^*$ is tame over $\mathcal{M}$ if and only if for every $a \in M^*$, the type $\mathrm{tp}(a|M)$ is definable. Marker and Steinhorn [3] generalized this to show that if $\mathcal{M} \preceq \mathcal{M}^*$ is tame over $\mathcal{M}$ then for every $a \in (M^*)^m$, the type $\mathrm{tp}(a|M)$ is definable. In particular if $(M, \leq) = (\mathbb{R}, \leq)$ then every type over $\mathbb{R}$ is definable. See [9] for geometric applications of this useful result. The original proof of Marker and Steinhorn uses a complicated inductive argument. Tressl [7] proved the Marker-Steinhorn for o-minimal expansions of real closed fields with a short

Date: July 19, 2018.

and clever argument using valuation theory and co-heirs that gives little idea as to the form of the defining formula of a type. Chernikov and Simon gave a proof using NIP-theoretic machinery [2]. We give a more constructive proof of the Marker-Steinhorn Theorem for o-minimal expansions of ordered groups. The crucial idea is to reduce the analysis of n-types to an analysis of cuts in definable linear orders. Our main tool is the following theorem of Ramakrishnan [5], which is closely related to earlier work of Onshuus-Steinhorn [4]. Let $\leq_{lex}$ be the lexiographic order on M^k.

Theorem 1.1. *Suppose $\mathcal{M}$ expands an ordered group. Then every definable linear order is definably isomorphic to a definable subset of some M^k equipped with the induced lexicographic order.*

Acknowledgments. We thank Matthias Aschenbrenner for suggesting the topic, for many useful discussions on the topic, and for finding a serious gap in the first version of the proof. We also thank David Marker for his comments on an earlier version of the proof.

2. Conventions

Throughout, $\mathcal{M}$ is an o-minimal expansion of an ordered abelian group, and $\mathcal{M} \preceq \mathcal{M}^* = (M^*, \ldots)$ is tame over $\mathcal{M}$. Unless said otherwise, "definable" means "definable, possibly with parameters," and the adjective "definable" applied to subsets of M^m or maps $A \to M^n$, $A \subseteq M^m$, will mean "definable in $\mathcal{M}$." The basic facts about o-minimal structures that we use may be found in [8]. We let m, n, k, l range over natural numbers. Given sets $A, B, C \subseteq A \times B$, and $a \in A$ we let

$$C_a := \{b \in B : (a, b) \in C\}.$$

If $A \subseteq M^m$ is a definable set, then A^* denotes the subset of $(M^*)^m$ defined in $\mathcal{M}^*$ by the same formula (since $\mathcal{M} \preceq \mathcal{M}^*$, this does not depend on the choice of a defining formula). Similarly, if $f : A \to M^n$ is a definable map, then $f^* : A^* \to (M^*)^n$ denotes the map whose graph is defined in $\mathcal{M}^*$ by the same formula as the graph of f. If $A \subseteq M^m$ is definable then $\dim(A)$ is the usual o-minimal dimension of A. Given a bounded definable $A \subseteq M^*$ we let $\mu(A)$ be the sum of the lengths of the components of A. We call $\mu(A)$ the *measure* of A. (Indeed, μ is a finitely additive measure on the collection of bounded $\mathcal{M}^*$-definable subsets of M^*). If $A \subseteq (M^*)^m \times M^*$ is such that every A_x is bounded then there is a definable $f : (M^*)^m \to M^*$ such that $f(x) = \mu(A_x)$.

3. Cuts in Definable Linear Orders

Throughout this section $(P, \leq_P)$ is a definable linear order and $P \subseteq M^m$.

Proposition 3.1. *If $V \subseteq P^*$ is $\mathcal{M}^*$ -definable and $W = V \cap P$ is a cut in P, then W is definable.*

The proof of this proposition is the most difficult part of this paper. The difficulty largely lies in the fact that V may not be a cut in P^*. We need the following three lemmas for Proposition 3.1. The first is an easy base case of the Marker-Steinhorn theorem, which we leave to the reader.

Lemma 3.2. *If $A \subseteq M^*$ is $\mathcal{M}^*$-definable then $A \cap M$ is definable.*

The second lemma follows easily from o-minimality, we leave the proof to the reader.

Lemma 3.3. *Let $0 < q < 1$ be a rational number. Suppose $I \subseteq M$ is a bounded interval and $J \subseteq I^*$ is $\mathcal{M}^*$-definable. If $\mu(J) \geq q\mu(I^*)$ then $J \cap I$ is nonempty. If $I \subseteq J$ then $\mu(J) \geq q\mu(I^*)$.*

Lemma 3.4. *Suppose $A \subseteq M^m$ is definable and B is an $\mathcal{M}^*$-definable subset of A^* such that A_x is either contained in or disjoint from B_x for all $x \in M^{m-1}$. Then there is an $\mathcal{M}^*$-definable $D \subseteq (M^*)^{m-1}$ such that*

$$D \cap M^{m-1} = \{x \in M^{m-1} : A_x \subseteq B_x\}.$$

Proof. Let $\{C_1, \ldots, C_n\}$ be a cell decomposition of A. Note that $(C_i)_x$ is either contained in or disjoint from B_x for all $x \in M^{m-1}$ and $1 \leq i \leq n$. If D_i is an $\mathcal{M}^*$-definable subset of $(M^*)^{m-1}$ such that

$$D_i \cap M^{m-1} = \{x \in M^{m-1} : (C_i)_x \subseteq B_x\} \quad \text{for } 1 \leq i \leq n,$$

then $D := D_1 \cap \ldots \cap D_n$ satisfies the conditions of the lemma. We therefore assume A is a cell. We now consider four cases. The first case is when A is the graph of a continuous definable $f : A' \to M$ on a cell $A' \subseteq M^{m-1}$. In this case we take D to be the set of $x \in (A')^*$ such that $f(x) \in B_x$. The second case is when

$$A = \{(x,t) \in M^{m-1} \times M : x \in A', f(x) < t < g(x)\}$$

for continuous definable $f, g : A' \to M$ on a cell $A' \subseteq M^{m-1}$ such that $f(x) < g(x)$ for all $x \in A'$. It follows from Lemma 3.3 that $A_x \subseteq B_x$ implies

$$\mu(B_x) \geq \frac{1}{2}\mu(A_x^*) = \frac{1}{2}[g^*(x) - f^*(x)] \quad \text{for all } x \in A'.$$

Lemma 3.3 also shows that for all $x \in A'$, if $\mu(B_x) \geq \frac{1}{2}\mu(A_x^*)$ then A_x and B_x intersect. We therefore take D to be the set of $x \in (A')^*$ such that $\mu(B_x) \geq \frac{1}{2}\mu(A_x^*)$. The third case is when

$$A = \{(x,y) \in M^{m-1} \times M : x \in A', y > f(x)\}$$

for continuous definable $f : A' \to M$ on a cell $A' \subseteq M^{m-1}$. If B_x contains A_x then B_x contains $\{y \in M : f(x) < y < f(x) + 1\}$. Conversely if B_x contains $\{y \in M : f(x) < y < f(x)+1\}$ then B_x intersects A_x and hence contains A_x by assumption. Thus B_x contains A_x if and only if it contains $\{y \in M : f(x) < y < f(x) + 1\}$. Reasoning as before we take D to be the set of $x \in (A')^*$ such that

$$\mu\left(A_x \cap \{y \in M^* : f(x) < y < f(x) + 1\}\right) \geq \frac{1}{2}.$$

The fourth case is when

$$A = \{(x,y) \in M^{m-1} \times M : x \in A', y < g(x)\}$$

for continuous definable $g : A' \to M$ on a cell $A' \subseteq M^{m-1}$. This case follows in the same way as the third case. □

We now prove Proposition 3.1

Proof. Applying Theorem 1.1 let $P' \subseteq M^k$ be definable, $\leq_{lex}$ be the restriction of the lexicographic order on M^k to P', and suppose $\iota : (P, \leq_P) \to (P', \leq_{lex})$ is a definable isomorphism of linear orders. It suffices to show $\iota(W) = \iota^*(V) \cap P'$ is definable. We therefore suppose $\leq_P$ is the restriction of the lexicographic order on M^m to P. We apply induction on m. If $m = 1$ then W is definable by Lemma 3.2. Suppose $m \geq 2$, let $\pi : P \to M^{m-1}$ be the projection onto the first $m-1$ coordinates, and let $Q = \pi(P)$. Note π is a monotone map $(P, \leq_P) \to (Q, \leq_{lex})$, it follows that $\pi(W)$ is a cut in $(Q, \leq_{lex})$. We consider two cases:

(1) $\pi(W)$ has a maximum q in $(Q, \leq_{lex})$,
(2) $\pi(W)$ does not have a maximum in $(Q, \leq_{lex})$.

We first treat case (1). The assumption implies that $\pi^{-1}(q) \cap W$ is upwards cofinal in W, so W is the downwards closure of $\pi^{-1}(q) \cap W$. Lemma 3.2 shows $\pi^{-1}(q) \cap W$ is definable, so W is definable.

We now treat case (2). If $p \in W$ then as $\pi(p)$ is not the maximal element of $\pi(W)$ it follows that $\pi(p') >_{lex} \pi(p)$ for some $p' \in W$, which implies $p' >_P q$ for all $q \in \pi^{-1}(p)$, as W is downwards closed we have $\pi^{-1}(p) \subseteq W$. Thus, for any $p \in Q$, if W intersects $\pi^{-1}(p)$ then W contains $\pi^{-1}(p)$. Note in particular that this implies $W = \pi^{-1}(\pi(W))$, so it suffices to show $\pi(W)$ is definable. Applying Lemma 3.4 we obtain an $\mathcal{M}^*$-definable $D \subseteq M^{m-1}$ such that $D \cap Q = \pi(W)$. As $\pi(W)$ is a cut in $(Q, \leq_{lex})$ the inductive hypothesis implies $\pi(W)$ is definable. □

The proof of Proposition 3.1 may be simplified by applying a result of Shelah, see [6] or [1]. This result, which holds for any NIP structure, implies that if $D \subseteq (M^*)^k$ is $\mathcal{M}^*$-definable and $\pi : (M^*)^k \to (M^*)^l$ is a coordinate projection then there is an $\mathcal{M}^*$-definable $E \subseteq (M^*)^l$ such that $\pi(D \cap M^k) = E \cap M^l$. Applying this result allows us to avoid the use of Lemma 3.4 and directly apply the inductive assumption to $\pi(W)$.

4. Proof of Marker-Steinhorn

Fix $b = (b_1, \ldots, b_k) \in (M^*)^k$. The following theorem shows $\operatorname{tp}(b|M)$ is definable.

Theorem 4.1. *If $A \subseteq M^l \times M^k$ is definable then $\{a \in M^l : (a, b) \in A^*\}$ is also definable.*

Proof. We apply induction on k. The base case $k = 1$ holds as all 1-types over M realized in M^* are definable. Suppose $k \geq 2$ and let $b' = (b_1, \ldots, b_{k-1})$. We declare $\dim(b|M)$ to be the minimal dimension of a definable $B \subseteq M^k$ such that $b \in B^*$. We first consider the case $\dim(b|M) < k$. Let $B \subseteq M^k$ be definable such that $b \in B^*$ and $\dim B < k$. Let $\{C_1, \ldots, C_n\}$ be a cell decomposition of B, let $1 \leq i \leq n$ be such that $b \in (C_i)^*$. After replacing B with C_i if necessary we suppose B is a cell. As $\dim B < k$ we suppose, after permuting coordinates if necessary, that

$$B = \{(a, t) \in M^{k-1} \times M : a \in B', t = f(a)\}$$

for a cell $B' \subseteq M^{k-1}$ and a continuous definable $f : B' \to M$. Note $b_k = f^*(b')$. Let E be the set of $(a, c) \in M^l \times M^{k-1}$ such that $(a, c, f(c)) \in A$. Given $a \in M^l$, we have $(a, b) \in A^*$ if and only if $(a, b') \in E^*$. Applying the inductive hypothesis to b' shows $\{a \in M^l : (a, b') \in E^*\}$ is definable. We therefore suppose $\dim(b|M) = k$.

Suppose $\{C_1, \ldots, C_n\}$ is a cell decomposition of A. It suffices to show that $\{a \in M^l : (a, b) \in (C_i)^*\}$ is definable for $1 \leq i \leq n$. We therefore suppose A is a cell. We suppose without loss of generality that $\{a \in M^l : (a, b) \in A^*\}$ is nonempty. As $\dim(b|M) = k$ it follows that $\dim(A_x) = k$ for some $x \in M^l$. As A is a cell it follows that $\dim(A_x) = k$ for all $x \in M^l$ such that $A_x \neq \emptyset$, so each A_x is an open cell. Then one of the following holds:

- $A = \{(a, c, t) \in M^l \times M^{k-1} \times M : (a, c) \in A', f(a, c) < t < g(a, c)\}$,
- $A = \{(a, c, t) \in M^l \times M^{k-1} \times M : (a, c) \in A', t < g(a, c)\}$,

- $A = \{(a, c, t) \in M^l \times M^{k-1} \times M : (a, c) \in A', f(a, c) < t\}$,

for a cell $A' \subseteq M^l \times M^{k-1}$ and continuous definable $f, g : A' \to M$. We only treat the third case as the previous two may be handled in the same way. In this case $(a, b) \in A^*$ if and only if $(a, b') \in (A')^*$ and $f^*(a, b') < b_k$. Let D be the set of $a \in M^l$ such that $(a, b') \in (A')^*$. An application of the inductive hypothesis shows D is definable. Let $\sim$ be the equivalence relation on D given by $e \sim d$ if and only if $f^*(e, b') = f^*(d, b')$. The inductive hypothesis shows $\sim$ is definable. Applying the elimination of imaginaries for o-minimal expansions of ordered groups we suppose $D/_\sim$ is a definable set P and let $\rho : D \to P$ be the quotient map. We put a relation $\trianglelefteq$ on D by declaring $e \trianglelefteq d$ if and only if $f^*(e, b') \leq f^*(d, b')$. The inductive hypothesis shows $\trianglelefteq$ is definable. It is easy to see that $\trianglelefteq$ is a quasi-order on D which pushes forward to a definable linear order on P under ρ. Abusing notation we let $\trianglelefteq$ be the push-forward of $\trianglelefteq$ to P. Let W be the set of $d \in P$ for which there is an $e \in D$ such that $\rho(e) = d$ and $f^*(e, b') < b_k$. Then

$$\{a \in M^l : (a, b) \in A^*\} = \{a \in M^l : [a \in D] \wedge [\rho(a) \in W]\}.$$

It is easy to see that W is a cut in $(P, \trianglelefteq)$. It follows by Proposition 3.1 that W is definable. □

References

[1] A. Chernikov and P. Simon. Externally definable sets and dependent pairs. *Israel J. Math.*, 194(1):409–425, 2013.

[2] A. Chernikov and P. Simon. Externally definable sets and dependent pairs II. *Trans. Amer. Math. Soc.*, 367(7):5217–5235, 2015.

[3] D. Marker and C. I. Steinhorn. Definable types in o-minimal theories. *J. Symbolic Logic*, 59(1):185–198, 1994.

[4] A. Onshuus and C. Steinhorn. On linearly ordered structures of finite rank. *J. Math. Log.*, 9(2):201–239, 2009.

[5] J. Ramakrishnan. Definable linear orders definably embed into lexicographic orders in o-minimal structures. *Proc. Amer. Math. Soc.*, 141(5):1809–1819, 2013.

[6] S. Shelah. Dependent first order theories, continued. *Israel J. Math.*, 173:1–60, 2009.

[7] M. Tressl. Valuation theoretic content of the Marker-Steinhorn theorem. *J. Symbolic Logic*, 69(1):91–93, 2004.

[8] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[9] L. van den Dries. Limit sets in o-minimal structures. In *Proceedings of the RAAG Summer School Lisbon 2003: O-minimal Structures*, pages 172–215. Lecture Notes in Real Algebraic and Analytic Geometry, Cuvillier Verlag, Gottingen, 2005.