EXPANSIONS OF THE REAL FIELD BY DISCRETE SUBGROUPS OF $\mathrm{Gl}_n(\mathbb{C})$

PHILIPP HIERONYMI, ERIK WALSBERG, AND SAMANTHA XU

Abstract. Let Γ be an infinite discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Then either $(\mathbb{R},<,+,\cdot,\Gamma)$ is interdefinable with $(\mathbb{R},<,+,\cdot,\lambda^{\mathbb{Z}})$ for some $\lambda \in \mathbb{R}$, or $(\mathbb{R},<,+,\cdot,\Gamma)$ defines the set of integers. When Γ is not virtually abelian, the second case holds.

1. Introduction

Let $\bar{\mathbb{R}} = (\mathbb{R},<,+,\cdot,0,1)$ be the real field. For $\lambda \in \mathbb{R}_{>0}$, set $\lambda^{\mathbb{Z}} := \{\lambda^m : m \in \mathbb{Z}\}$. Throughout this paper Γ denotes a discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$, and G denotes a subgroup of $\mathrm{Gl}_n(\mathbb{C})$. We identify the set $\mathrm{M}_n(\mathbb{C})$ of n-by-n complex matrices with $\mathbb{C}^{n^2}$ and identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Our main result is the following classfication of expansions of $\bar{\mathbb{R}}$ by a discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$.

Theorem A. *Let Γ be an infinite discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Then either*

- $(\mathbb{R},\Gamma)$ *defines* $\mathbb{Z}$ *or*
- *there is $\lambda \in \mathbb{R}_{>0}$ such that $(\bar{\mathbb{R}},\Gamma)$ is interdefinable with $(\bar{\mathbb{R}},\lambda^{\mathbb{Z}})$.*

If Γ is not virtually abelian, then $(\bar{\mathbb{R}},\Gamma)$ defines $\mathbb{Z}$.

By Hieronymi [11, Theorem 1.3], the structure $(\bar{\mathbb{R}},\lambda^{\mathbb{Z}},\mu^{\mathbb{Z}})$ defines $\mathbb{Z}$ whenever $\log_\lambda \mu \notin \mathbb{Q}$, and is interdefinable with $(\bar{\mathbb{R}},\lambda^{\mathbb{Z}})$ otherwise. Therefore Theorem A extends immediately to expansions of $\bar{\mathbb{R}}$ by multiple discrete subgroups of $\mathrm{Gl}_n(\mathbb{C})$.

Corollary A. *Let $\mathcal{G}$ be a collection of infinite discrete subgroups of various $\mathrm{Gl}_n(\mathbb{C})$. Then either*

- $(\bar{\mathbb{R}},(\Gamma)_{\Gamma\in\mathcal{G}})$ *defines* $\mathbb{Z}$ *or*
- *there is $\lambda \in \mathbb{R}_{>0}$ such that $(\bar{\mathbb{R}},(\Gamma)_{\Gamma\in\mathcal{G}})$ is interdefinable with $(\bar{\mathbb{R}},\lambda^{\mathbb{Z}})$.*

The dichotomies in Theorem A and Corollary A are arguably as strong as they can be. An expansion of the real field that defines $\mathbb{Z}$, has not only an undecidable theory, but also defines every real projective set in sense of descriptive set theory (see Kechris [16, 37.6]). From a model-theoretic/geometric point of view such a structure is a wild as can be. On the other hand, by van den Dries [4] the structure $(\bar{\mathbb{R}},\lambda^{\mathbb{Z}})$ has a decidable theory whenever λ is recursive, and admits quantifier-elimination in a suitably extended language. It satisfies combinatorical model-theoretic tameness conditions such as NIP and distality (see [9, 15]). Furthermore, it follows

Date: September 6, 2018.

This is a preprint version. Later versions might contain significant changes. The first author was partially supported by NSF grant DMS-1654725. The second author was partially supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 291111/ MODAG .

from these results that every subset of $\mathbb{R}^n$ definable in $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ is a boolean combination of open sets, and thus $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ defines only sets on the lowest level of the Borel hierarchy. See Miller [18] for more on tameness in expansions of the real field.

Our proof of Theorem A relies crucially on the following two criteria for the definability of $\mathbb{Z}$ in expansions of the real field.

Fact 1.1. *Suppose $D \subseteq \mathbb{R}^k$ is discrete.*

(1) *If $(\bar{\mathbb{R}}, D)$ defines a subset of $\mathbb{R}$ that is dense and co-dense in a nonempty open interval, then $(\bar{\mathbb{R}}, D)$ defines $\mathbb{Z}$.*
(2) *If D has positive Assouad dimension, then $(\bar{\mathbb{R}}, D)$ defines $\mathbb{Z}$.*

The first statement is [12, Theorem E], a fundamental theorem on first-order expansions of $\bar{\mathbb{R}}$, and the second claim is proven using the first in Hieronymi and Miller [14, Theorem A]. We recall the definition of Assouad dimension in Section 5. This important metric dimension bounds more familiar metric dimensions (such as Hausdorff and Minkowski dimension) from above. We refer to [14] for a more detailed discussion of Assouad dimension and its relevance to definability theory.

The outline of our proof of Theorem A is as follows. Let Γ be a discrete, infinite subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Using Fact 1.1(1), we first show that $(\bar{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ whenever Γ contains a non-diagonalizable matrix. It follows from a theorem of Mal'tsev that $(\bar{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ when Γ is virtually solvable and not virtually abelian. In the case that Γ is not virtually solvable, we prove using Tits' alternative that Γ has positive Assouad dimension, and hence $(\bar{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ by Fact 1.1(2). We conclude the proof of Theorem A by proving that whenever Γ is virtually abelian and $(\bar{\mathbb{R}}, \Gamma)$ does not define $\mathbb{Z}$, then $(\bar{\mathbb{R}}, \Gamma)$ is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ for some $\lambda \in \mathbb{R}_{>0}$. Along the way we give (Lemma 3.4) an elementary proof showing that a torsion free non abelian nilpotent subgroup of $\mathrm{Gl}_n(\mathbb{C})$ has a non-diagonalizable element. As every finitely generated subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is either virtually nilpotent or has exponential growth, this yields a more direct proof of Theorem A in the case when Γ is finitely generated.

We want to make an extra comment about the case when Γ is a discrete, virtually solvable, and not virtually abelian subgroup of $\mathrm{Gl}_n(\mathbb{C})$. The **Novosibirsk theorem** [22] of Noskov (following work of Mal'stev, Ershov, and Romanovskii) shows that a finitely generated, virtually solvable and non-virtually abelian group interprets $(\mathbb{Z}, +, \cdot)$. It trivially follows that if G is finitely generated, virtually solvable, and non-virtually abelian, then $(\bar{\mathbb{R}}, G)$ interprets $(\mathbb{Z}, +, \cdot)$. However, it does not directly follow that $(\bar{\mathbb{R}}, G)$ defines $\mathbb{Z}$. We use an entirely different method below to show that if G is in addition discrete, then $(\bar{\mathbb{R}}, G)$ defines $\mathbb{Z}$. Our method also applies when G is not finitely generated, but relies crucially on the discreteness of G.

This paper is by no means the first paper to study expansions of the real field by subgroups of $\mathrm{Gl}_n(\mathbb{C})$. Indeed, there is a large body of work on this subject, often not explicitly mentioning $\mathrm{Gl}_n(\mathbb{C})$. Because we see this paper as part of a larger investigation, we survey some of the earlier results and state a conjecture. It is convenient to consider three disctinct classes of such expansion. By Miller and Speissegger [20] every first-order expansion $\mathcal{R}$ of $\bar{\mathbb{R}}$ satisfies at least one of the following:

(1) $\mathcal{R}$ is o-minimal,
(2) $\mathcal{R}$ defines an infinite discrete subset of $\mathbb{R}$,
(3) $\mathcal{R}$ defines a dense and co-dense subset of $\mathbb{R}$.

The **open core** $\mathcal{R}^\circ$ of $\mathcal{R}$ is the expansion of $(\mathbb{R}, <)$ generated by all open $\mathcal{R}$-definable subsets of all $\mathbb{R}^k$. By [20], if $\mathcal{R}$ does not satisfy (2), then $\mathcal{R}^\circ$ is o-minimal.

The case when $\mathcal{R}$ is o-minimal, is largely understood. Wilkie's famous theorem [28] that $(\bar{\mathbb{R}}, \exp)$ is o-minimal is crucial. This shows the expansion of $\bar{\mathbb{R}}$ by the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & \lambda^t & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

is o-minimal for $\lambda \in \mathbb{R}_{>0}$, and so is the expansion of $\bar{\mathbb{R}}$ by any subgroup of the form

$$\left\{ \begin{pmatrix} t^s & 0 \\ 0 & t^r \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}$$

for $s, r \in \mathbb{R}_{>0}$. Indeed, by Peterzil, Pillary, and Starchenko [24], whenever an expansion $(\bar{\mathbb{R}}, G)$ by a subgroup G of $\mathrm{Gl}_n(\mathbb{R})$ is o-minimal, then G is already definable in $(\bar{\mathbb{R}}, \exp)$. Futhermore, note that by a classical theorem of Tannaka and Chevalley [3] every compact subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is the group of real points on an algebraic group defined over $\mathbb{R}$. Thus every compact subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is $\bar{\mathbb{R}}$-definable, and therefore the case of expansions by compact subgroups of $\mathrm{Gl}_n(\mathbb{C})$ is understood as well.

We now consider the case when infinite discrete sets are definable. Corollary A for discrete subgroups of $\mathbb{C}^\times$ follows easily from the proof of [11, Theorem 1.6]. While Corollary A handles the case of expansions by discrete subgroups of $\mathrm{Gl}_n(\mathbb{C})$, there are examples of subgroups of $\mathrm{Gl}_n(\mathbb{C})$ that define infinite discrete sets, but fail the conclusion of Theorem A. Given $\alpha \in \mathbb{R}^\times$ the logarithmic spiral

$$S_\alpha = \{(\exp(t)\sin(\alpha t), \exp(t)\cos(\alpha t)) : t \in \mathbb{R}\}$$

is a subgroup of $\mathbb{C}^\times$. Let $\mathfrak{s}$ and $\mathfrak{e}$ be the restrictions of sin and exp to $[0, 2\pi]$, respectively. Then $(\bar{\mathbb{R}}, S_\alpha)$ is a reduct of $(\bar{\mathbb{R}}, \mathfrak{s}, \mathfrak{e}, \lambda^{\mathbb{Z}})$ when $\lambda = \exp(2\pi\alpha)$, as was first observed by Miller and Speissegger. As $(\bar{\mathbb{R}}, \mathfrak{s}, \mathfrak{u})$ is o-minimal with field of exponents $\mathbb{Q}$, the structure $(\bar{\mathbb{R}}, S_\alpha)$ is d-minimal[1] by Miller [18, Theorem 3.4.2] and thus does not define $\mathbb{Z}$. It can be checked that $(\bar{\mathbb{R}}, S_\alpha)$ defines a analytic function that is not semi-algebraic[2], and thus is not interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ for any $\lambda \in \mathbb{R}_{>0}$.

Most work in the case of expansions that define dense and co-dense sets, concerns expansions by finite rank subgroups of $\mathbb{C}^\times$ (see introduction of [2] for a thorough discussion of expansions by subgroups of $\mathbb{C}^\times$). In [5] van den Dries and Günaydın

[1]A expansion $\mathcal{R}$ of $\bar{\mathbb{R}}$ is **d-minimal** if every definable unary set in every model of the theory of $\mathcal{R}$ is a union of an open set and finitely many discrete sets.

[2]By induction on the complexity of terms it follows easily from [Theorem II, vdD] that the definable functions in $(\overline{R}, \lambda^{\mathbb{Z}}$ are given piecewise by a finite compositions of $x \mapsto \max\Big(\{0\} \cup (\lambda^{\mathbb{Z}} \cap [-\infty, x])\Big)$ and functions definable in $\overline{R}$. From this one can deduce that every definable function in this structure is piecewise semi-algebraic.

showed that an expansion of $\bar{\mathbb{R}}$ by a finitely generated dense subgroup of $(\mathbb{R}_{>0}, \cdot)$ admits quantifier-elimination in a suitably extend language. Günaydın [8] and Belegradek and Zilber [1] proved similar results for the expansion of $\bar{\mathbb{R}}$ by a dense finite rank subgroup of the unit circle $\mathbb{U} := \{a \in \mathbb{C}^\times : |a| = 1\}$. This covers the case when G is the group of roots of unity. In all these cases the open core of the resulting expansion is interdefinable with $\bar{\mathbb{R}}$. This does not always have to be the case. In Caulfield [**?**] studies expansions by subgroups of $\mathbb{C}^\times$ of the form

$$\{\lambda^k \exp(i\alpha l) : k, l \in \mathbb{Z}\} \quad \text{where } \lambda \in \mathbb{R}_{>0} \text{ and } \alpha \in \mathbb{R} \setminus \pi\mathbb{Q}.$$

Such an expansions obviously defines a dense and co-dense subset of $\mathbb{R}$, but by [**?**] its open core is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$. Futhermore, even if the open core is o-minimal, it does not have to be interdefinable with $\bar{\mathbb{R}}$. By [13] there is a co-countable subset Λ of $\mathbb{R}_{>0}$ such that if $r \in \Lambda$ and H is a finitely generated dense subgroup of $(\mathbb{R}_{>0}, \cdot)$ contained in the algebraic closure of $\mathbb{Q}(r)$, then the open core of the expansion of $\bar{\mathbb{R}}$ by the subgroup

$$\left\{ \begin{pmatrix} t & 0 \\ 0 & t^r \end{pmatrix} : t \in H \right\}$$

is interdefinable with the expansion of $\bar{\mathbb{R}}$ by the power function $t \mapsto t^r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$.

All these previous results suggest that the next class of subgroups of $\mathrm{Gl}_n(\mathbb{C})$ for which we can hope to prove a classification comparable to Theorem A, is the class of finitely generated subgroups. Here the following conjecture seems natural, but most likely very hard to prove. Let $\bar{\mathbb{R}}_{\mathrm{Pow}}$ be the expansion of $\bar{\mathbb{R}}$ by all power functions $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$ of the form $t \mapsto t^r$ for $r \in \mathbb{R}^\times$.

Conjecture. *Let G be a finitely generated subgroup of $\mathrm{Gl}_n(\mathbb{C})$ such that $(\bar{\mathbb{R}}, G)$ does not define $\mathbb{Z}$. Then the open core of $(\bar{\mathbb{R}}, G)$ is a reduct of $\bar{\mathbb{R}}_{Pow}$ or of $(\bar{\mathbb{R}}, S_\alpha)$ for some $\alpha \in \mathbb{R}_{>0}$.*

Even when the statement "$(\bar{\mathbb{R}}, G)$ does not define $\mathbb{Z}$" is replaced by "$(\bar{\mathbb{R}}, G)$ does not interpret $(\mathbb{Z}, +, \cdot)$", the conjecture is open. However, this weaker conjecture might be easier to prove, because the Novosibirsk theorem can be used to rule out the case when G is virtually solvable and non-virtually abelian. It is worth pointing out that Caulfield conjectured that when G is assumed to be a subgroup of $\mathbb{C}^\times$, then the open core $(\bar{\mathbb{R}}, G)$ is either $\bar{\mathbb{R}}$ or a reduct of $(\bar{\mathbb{R}}, S_\alpha)$ for some $\alpha \in \mathbb{R}_{>0}$. See [**?**, 2] for progress towards this later conjecture.

2. Notation and Conventions

Throughout m, n range over $\mathbb{N}$ and k, l range over $\mathbb{Z}$, G is a subgroup of $\mathrm{Gl}_n(\mathbb{C})$, and Γ is a discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Let $\bar{\mathbb{R}}_\Gamma$ be the expansion of $\bar{\mathbb{R}}$ by a $(2n)^2$-ary predicate defining Γ. We set $\bar{\mathbb{R}}_\lambda := \bar{\mathbb{R}}_{\lambda^{\mathbb{Z}}}$. A subset of $\mathbb{R}^k$ is **discrete** if every point is isolated. We let $\mathrm{UT}_n(\mathbb{C})$ be the group of n-by-n upper triangular matrices, $\mathrm{D}_n(\mathbb{C})$ be the group of n-by-n diagonal matrices, and $\mathbb{U}$ be the multiplicative group of complex numbers with norm one.

All structures considered are first-order, "definable" means "definable, possibly with parameters". Two expansions of $(\mathbb{R}, <)$ are **interdefinable** if they define the same subsets of $\mathbb{R}^k$ for all k. If P is a propety of groups then a group H is **virtually P** if there is finite index subgroup H' of H that is P.

3. Linear Groups

We gather some general facts on groups. Throughout this section H is a finitely generated group with a symmetric set S of generators. Let S_m be the set of m-fold products of elements of S for all m. If S' is another symmetric set of generators then there is a constant $k \geq 1$ such that

$$k^{-1}|S_m| \leq |S'_m| \leq k|S_m| \quad \text{for all } m.$$

Thus the growth rate of $m \mapsto |S_m|$ is an invariant of H. We say H has **exponential growth** if there is a $C \geq 1$ such that $|S_m| \geq C^m$ for all m and H has **polynomial growth** there are $k, t \in \mathbb{R}_{>0}$ such that $|S_m| \leq tm^k$ for all m. Note finitely generated non-abelian free groups are of exponential growth. Gromov's theorem [7] says H has polynomial growth if and only if it is virtually nilpotent. Gromov's theorem for subgroups of $\mathrm{Gl}_n(\mathbb{C})$ is less difficult and may be proven using the following two theorems:

Fact 3.1. *If G does not contain a non-abelian free subgroup, then G is virtually solvable.*

Fact 3.1 is **Tits' alternative** [26]. Fact 3.2 is due to Milnor [21] and Wolf [29].

Fact 3.2. *Suppose H is virtually solvable. Then H either has exponential or polynomial growth. If the latter case holds then H is virtually nilpotent.*

Note Fact 3.1 and Fact 3.2 imply every finitely generated subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is of polynomial or exponential growth. This dichotomy famously does not hold for finitely generated groups in general, see for example [6].

The **Heisenberg group** $\mathbb{H}$ is presented by generators a, b, c and relations

$$[a, b] = c, \quad ac = ca, \quad bc = cb.$$

The following fact is folklore; we include a proof for the reader.

Fact 3.3. *Let E be a nilpotent, torsion-free, and non-abelian group. Then there is a subgroup of E isomorphic to $\mathbb{H}$.*

Proof. Let e be the identity element of E. We define the lower central series $(E_k)_{k\in\mathbb{N}}$ of E by declaring $E_0 = E$ and $E_k = [E_{k-1}, E]$ for $k \geq 1$. Nilpotency means there is an m such that $E_m \neq \{e\}$ and $[E_m, E] = \{e\}$. Moreover $m \geq 1$ as E is not abelian.

On one hand, $[E_{m-1}, E] = E_m \neq \{e\}$ and so E_{m-1} is not contained in $Z(E)$. Thus, there exists $a \in E_{m-1} \setminus Z(E)$ and $b \in E_m$ that does not commute with a. On the other hand, $[E_m, E] = \{e\}$ implies E_m is contained in the center $Z(E)$ of E and is thus abelian. So, $c := [a, b]$ is an element of $Z(E)$ and commutes with both a and b.

Finally, a, b, c have infinite order because E is torsion-free. So, a, b, c generate a subgroup of E isomorphic to the Heisenberg group. ☐

3.1. Non-diagonalizable elements.

We show certain linear groups necessarily contain non-diagonalizable elements.

Lemma 3.4. *If G is nilpotent, torsion-free, and not abelian, then G contains a non-diagonalizable element.*

Lemma 3.4 follows from Fact 3.3 above and Lemma 3.5 below.

Lemma 3.5. *Suppose $a, b, c \in Gl_n(\mathbb{C})$ satisfy*

$$[a, b] = c, \quad ac = ca, \quad bc = cb,$$

and c is not torsion. Then either a or c is not diagonalizable.

Proof. Suppose a, c are both diagonalizable. As a, c commute, they are simultaneously diagonalizable and share a basis $\mathfrak{B}$ of eigenvectors. As c is not torsion, there is $\lambda_c \in \mathbb{C}^\times$ which is not a root of unity and $v \in \mathfrak{B}$ such that $cv = \lambda_c v$. Let $\lambda_a \in \mathbb{C}^\times$ be such that $av = \lambda_a v$.

By way of contradiction, we will show $a(b^k v) = (\lambda_a \lambda_c^k)(b^k v)$ for all $k \geq 1$. As λ_c is not a root of unity, this implies a has infinitely many eigenvalues, which is impossible for an $n \times n$ matrix. The base case holds as

$$a(bv) = bacv = (\lambda_a \lambda_c)(bv).$$

Let $k \geq 2$ and suppose $a(b^{k-1}v) = (\lambda_a \lambda_c^{k-1})(b^{k-1}v)$. As c commutes with b,

$$a(b^k v) = ab(b^{k-1}v) = bac(b^{k-1}v) = bab^{k-1}cv = (\lambda_c)(bab^{k-1}v).$$

Applying the inductive assumption,

$$(\lambda_c)(bab^{k-1}v) = (\lambda_c)b(\lambda_a \lambda_c^{k-1} b^{k-1} v) = (\lambda_a \lambda_c^k)(b^k v).$$

□

We now prove a slight weakening of Lemma 3.4 for solvable groups. Recall $a \in \mathrm{Gl}_n(\mathbb{C})$ is **unipotent** if some conjugate of a is upper triangular with every diagonal entry equal to one. The only diagonalizable unipotent matrix is the identity. We recall a theorem of Mal'tsev [17].

Fact 3.6. *Suppose G is solvable. Then there is a finite index subgroup G' of G such that G' is conjugate to a subgroup of $UT_n(\mathbb{C})$.*

We now derive an easy corollary from Fact 3.6

Lemma 3.7. *Suppose G is solvable and not virtually abelian. Then G contains a non-diagonalizable element.*

Proof. Suppose every element of G is diagonalizable. After applying Fact 3.6 and making a change of basis if necessary we suppose $G' = G \cap \mathrm{UT}_n(\mathbb{C})$ has finite index in G. Let $\rho : \mathrm{UT}_n(\mathbb{C}) \to \mathrm{D}_n(\mathbb{C})$ be the natural quotient map; that is the restriction to the diagonal. Every element of the kernel of ρ is unipotent. Thus the restriction of ρ to G' is injective, and so G' is abelian. □

4. Non-diagonalizable matrices

Lemma 4.1. *Suppose G contains a non-diagonalizable matrix. Then there is a rational function h on $Gl_n(\mathbb{C}) \times Gl_n(\mathbb{C})$ such that $h(G \times G) \subseteq \mathbb{C}$ is dense in $\mathbb{R}_{>0}$.*

Proof. Suppose $a \in G$ is non-diagonalizable. Let $b \in \mathrm{Gl}_n(\mathbb{C})$ be such that bab^{-1} is in Jordan form, i.e.

$$bab^{-1} = \begin{pmatrix} A_1 & O & \dots & O \\ O & A_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_l \end{pmatrix}$$

where each A_i is a Jordan block and each O is a zero matrix of the appropriate dimensions. We have

$$ba^kb^{-1} = \begin{pmatrix} A_1^k & O & \dots & O \\ O & A_2^k & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_l^k \end{pmatrix} \quad \text{for all } k.$$

As a is not diagonalizable, A_k has more then one entry for some k. We suppose A_1 is m-by-m with $m \geq 2$. For some $\lambda \in \mathbb{C}^\times$ we have

$$A_1 = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

It is well-known and easy to show by induction that for every $k \geq 1$:

$$A_1^k = \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \binom{k}{3}\lambda^{k-3} & \binom{k}{4}\lambda^{k-4} & \dots & \binom{k}{m}\lambda^{k-m+1} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \binom{k}{3}\lambda^{k-3} & \dots & \binom{k}{m-1}\lambda^{k-m+2} \\ 0 & 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \dots & \binom{k}{m-2}\lambda^{k-m+3} \\ 0 & 0 & 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \dots & \binom{k}{m-3}\lambda^{k-m+4} \\ 0 & 0 & 0 & 0 & \lambda^k & \dots & \binom{k}{m-4}\lambda^{k-m+5} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & 0 & \dots & \dots & 0 & \lambda^k \end{pmatrix}.$$

Let g_{ij} be the (i,j)-entry of $g \in \mathrm{Gl}_n(\mathbb{C})$. Thus, for each $k \geq 1$,

$$(ba^kb^{-1})_{01} = k\lambda^{k-1} \quad \text{and} \quad (ba^kb^{-1})_{11} = \lambda^k.$$

We define a rational function h' on $\mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathbb{C})$ by declaring

$$h'(g,g') := \frac{g_{01}g'_{11}}{g'_{01}g_{11}}$$

for all $g, g' \in \mathrm{Gl}_n(\mathbb{C})$ such that $g_{11}, g'_{01} \neq 0$. Then define h by declaring

$$h(g,g') := h'(bgb^{-1}, bg'b^{-1})$$

We have

$$h(a^i, a^j) = \frac{(i\lambda^{i-1})(\lambda^j)}{(j\lambda^{j-1})(\lambda^i)} = \frac{i}{j} \quad \text{for all } i,j \geq 1.$$

Thus $\mathbb{Q}_{>0}$ is a subset of the image of $G \times G$ under h. □

Corollary 4.2. *If Γ contains a non-diagonalizable matrix, then $\bar{\mathbb{R}}_\Gamma$ defines $\mathbb{Z}$. In particular, if Γ is either*

- *solvable and not virtually abelian, or*

- *torsion-free, nilpotent and non-abelian,*

then $\bar{\mathbb{R}}_\Gamma$ *defines* $\mathbb{Z}$.

Proof. Applying Lemma 4.1, suppose h is a rational function on $\mathrm{Gl}_n(\mathbb{C}) \times \mathrm{Gl}_n(\mathbb{C})$ such that the image of $\Gamma \times \Gamma$ under h is dense in $\mathbb{R}_{>0}$. Note Γ is countable as Γ is discrete. It follows that the image of $\Gamma \times \Gamma$ under any function is co-dense in $\mathbb{R}_{>0}$. Fact 1.1(1) implies that $\bar{\mathbb{R}}_\Gamma$ defines $\mathbb{Z}$. The second claim follows from the first by applying Lemma 3.4 and Lemma 3.7. □

Corollary 4.3. *If* $a \in Gl_n(\mathbb{C})$ *is non-diagonalizable, then* $(\bar{\mathbb{R}}, \{a^k : k \in \mathbb{Z}\})$ *defines* $\mathbb{Z}$.

Proof. Set $G := \{a^k : k \in \mathbb{Z}\}$. The proof of Lemma 4.1 shows that in this case $\mathbb{Q}_{>0}$ is the intersection of $h(G \times G)$ and $\mathbb{R}_{>0}$. Thus the corollary follows by Julia Robinson's classical theorem of definability of $\mathbb{Z}$ in $(\mathbb{Q}, +, \cdot)$ in [25]. □

5. The case of exponential growth

We recall the **Assouad dimension** of a metric space (X, d). See Heinonen [10] for more information. The Assouad dimension of a subset Y of $\mathbb{R}^k$ is the Assouad dimension of Y equipped with the euclidean metric induced from $\mathbb{R}^k$.

Suppose $A \subseteq X$ has at least two elements. Then A is **δ-separated** for $\delta \in \mathbb{R}_{>0}$ if $d(a, b) \geq \delta$ for all distinct $a, b \in A$, and A is **seperated** if A is δ-seperated for some $\delta > 0$. Let $\mathcal{S}(A) \in \mathbb{R}$ be the supremum of all $\delta \geq 0$ for which A is δ-seperated. Let $\mathcal{D}(A)$ be the **diameter** of A; that is the infimum of all $\delta \in \mathbb{R} \cup \{\infty\}$ such that $d(a, b) < \delta$ for all $a, b \in A$, and A is **bounded** if $\mathcal{D}(A) < \infty$. Note $\mathcal{S}(A) \leq \mathcal{D}(A)$. The **Assouad dimension** of (X, d) is the infimum of the set of $\beta \in \mathbb{R}_{>0}$ for which there is a $C > 0$ such that

$$|A| \leq C \left(\frac{\mathcal{D}(A)}{\mathcal{S}(A)} \right)^\beta \text{ for all bounded and separated } A \subseteq X.$$

The proof of Fact 5.1 is an elementary computation which we leave to the reader.

Fact 5.1. *Suppose there is a sequence* $\{A_m\}_{m \in \mathbb{N}}$ *of bounded separated subsets of* X *with cardinality at least two, and* $B, C, t > 1$ *are such that*

$$|A_m| \geq C^m \quad \text{and} \quad \frac{\mathcal{D}(A_m)}{\mathcal{S}(A_m)} \leq tB^m \quad \text{for all } m$$

then (X, d) *has positive Assouad dimension.*

Let $|v|$ be the usual euclidean norm of $v \in \mathbb{C}^n$. Given $g \in \mathrm{M}_n(\mathbb{C})$ we let

$$\|g\| = \inf\{t \in \mathbb{R}_{>0} : |gv| \leq t|v| \text{ for all } v \in \mathbb{C}^n\}$$

be the operator norm of g. Then $\|\|$ is a linear norm on $\mathrm{M}_n(\mathbb{C})$ and satisfies $\|gh\| \leq \|g\|\|h\|$ for all $g, h \in \mathrm{M}_n(\mathbb{C})$. As any two linear norms on $\mathrm{M}_n(\mathbb{C})$ are bi-Lipschitz equivalent the metric induced by $\|\|$ is bi-Lipschitz equivalent to the usual euclidean metric on $\mathbb{R}^{n^2}$.

Proposition 5.2. *Suppose* Γ *contains a finitely generated subgroup* Γ' *of exponential growth. Then* Γ *has positive Assouad dimension.*

Proof. Because Assouad dimension is a bi-Lipschitz invariant (see [10]), it suffices to show that Γ has positive Assouad dimension with respect to the metric induced by $\|\|$. We let I be the n-by-n identity matrix. Let S be a symmetric generating set of Γ', and let S_m be the set of m-fold products of elements of S for $m \geq 2$. Set

$$B := \max\{\|g\| : g \in S\} \quad \text{and} \quad D := \min\{\|g - I\| : g \in \Gamma\}.$$

Note that $D > 0$, as Γ is discrete, and that $B > 0$, as $\Gamma \neq \{I\}$. Induction shows that $\|g\| \leq B^m$ when $g \in S_m$. The triangle inequality directly yields $\mathcal{D}(S_m) \leq 2B^m$. Each S_m is symmetric as S is symmetric. Therefore $\|g^{-1}\| \leq B^m$ for all $g \in S_m$. Let $g, h \in \Gamma$. We have

$$\|I - g^{-1}h\| \leq \|g^{-1}\|\|g - h\|.$$

Equivalently,

$$\frac{\|I - g^{-1}h\|}{\|g^{-1}\|} \leq \|g - h\|.$$

Suppose $g, h \in S_m$ are distinct. Then $g^{-1}h \neq I$, and hence $\|I - g^{-1}h\| \geq D$. So

$$\|g - h\| \geq \frac{\|I - g^{-1}h\|}{\|g^{-1}\|} \geq \frac{D}{B^m}.$$

Hence $\mathcal{S}(S_m) \geq D/B^m$. Thus

$$\frac{\mathcal{D}(S_m)}{\mathcal{S}(S_m)} \leq \frac{2B^m}{D/B^m} = \frac{2}{D}B^{2m}.$$

As Γ' has exponential growth, there is a $C > 0$ such that $|S_m| \geq C^m$ for all m. An application of Fact 5.1 shows that Γ has positive Assouad dimension. □

Proposition 5.3. *Suppose Γ is not virtually abelian. Then $\bar{\mathbb{R}}_\Gamma$ defines $\mathbb{Z}$.*

Proof. By Corollary 4.2, we can assume that Γ is solvable. Thus by Fact 3.1, the group Γ contains a non-abelian free subgroup. Therefore Γ has positive Assouad dimension by Proposition 5.2. We conclude that $\bar{\mathbb{R}}_\Gamma$ defines $\mathbb{Z}$ by Fact 1.1(2). □

6. The virtually abelian case

We first reduce the virtually abelian case to the abelian case.

Lemma 6.1. *Suppose G is virtually abelian and every element of G is diagonalizable. Then there is a finite index abelian subgroup G' of G such that $(\bar{\mathbb{R}}, G)$ and $(\bar{\mathbb{R}}, G')$ are interdefinable.*

Proof. Let G'' be a finite index abelian subgroup of G. As every element of G'' is diagonalizable, G'' is simultaneously diagonalizable. Fix $g \in \mathrm{Gl}_n(\mathbb{C})$ such that gag^{-1} is diagonal for all $a \in G''$. Let G' be the set of $a \in G$ such that gag^{-1} is diagonal, i.e. G' is the intersection of G and $g^{-1}\mathrm{D}_n(\mathbb{C})g$. Then G' is abelian, $(\bar{\mathbb{R}}, G)$-definable, and is of finite index in G as $G'' \subseteq G'$. Because G' has finite index in G, we have

$$G = g_1G' \cup \ldots \cup g_mG' \text{ for some } g_1, \ldots, g_m \in G.$$

So G is $(\bar{\mathbb{R}}, G')$-definable. □

Proposition 6.2 finishes the proof of Theorem A.

Proposition 6.2. *Suppose Γ is abelian and $\bar{\mathbb{R}}_\Gamma$ does not define $\mathbb{Z}$. Then there is $\lambda \in \mathbb{R}_{>0}$ such that $\bar{\mathbb{R}}_\Gamma$ is interdefinable with $\bar{\mathbb{R}}_\lambda$.*

Let $\mathfrak{u} : \mathbb{C}^\times \to \mathbb{U}$ be the argument map and $|\ | : \mathbb{C}^\times \to \mathbb{R}_{>0}$ be the absolute value map. Thus $z = \mathfrak{u}(z)|z|$ for all $z \in \mathbb{C}^\times$. Let $\mathbb{U}_m$ be the group of mth roots of unity for all $m \geq 1$. In the following proof of Proposition 6.2 we will use the immediate corollary of [11, Theorem 1.3] that the structure $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ defines $\mathbb{Z}$ whenever $\log_\lambda \mu \notin \mathbb{Q}$, and is is interdefinable with $(\bar{\mathbb{R}}, \lambda^{\mathbb{Z}})$ otherwise.

Proof. Fact 1.1(1) implies every countable $\bar{\mathbb{R}}_\Gamma$-definable subset of $\mathbb{R}$ is nowhere dense. It follows that every $\bar{\mathbb{R}}_\Gamma$-definable countable subgroup of $\mathbb{U}$ is finite and every $\bar{\mathbb{R}}_\Gamma$-definable countable subgroup of $(\mathbb{R}_{>0}, \cdot)$ is of the form $\lambda^{\mathbb{Z}}$ for some $\lambda \in \mathbb{R}_{>0}$.

Every element of Γ is diagonalizable by Corollary 4.2. Thus Γ is simultaneously diagonalizable. After making a change of basis we suppose Γ is a subgroup of $\mathrm{D}_n(\mathbb{C})$. We identify $\mathrm{D}_n(\mathbb{C})$ with $(\mathbb{C}^\times)^n$. Let Γ_i be the image of Γ under the projection $(\mathbb{C}^\times)^n \to \mathbb{C}^\times$ onto the ith cordinate for $1 \leq i \leq n$.

Each $\mathfrak{u}(\Gamma_i)$ is finite. Fix an m such that $\mathfrak{u}(\Gamma_i)$ is a subgroup of $\mathbb{U}_m$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, $|\Gamma_i|$ is a discrete subgroup of $\mathbb{R}_{>0}$ and is thus equal to $\alpha_i^{\mathbb{Z}}$ for some $\alpha_i \in \mathbb{R}_{>0}$. By [11, Theorem 1.3] each α_i is a rational power of α_1. Let $\lambda \in \mathbb{R}_{>0}$ be a rational power of α_1 such that each α_i is an integer power of λ. We show $\bar{\mathbb{R}}_\Gamma$ and $\bar{\mathbb{R}}_\lambda$ are interdefinable. Note that $\lambda^{\mathbb{Z}}$ is $\bar{\mathbb{R}}_\Gamma$-definable; so it suffices to show Γ is $\bar{\mathbb{R}}_\lambda$-definable.

Every element of Γ_i is of the form $\sigma\lambda^k$ for some $\sigma \in \mathbb{U}_m$ and $k \in \mathbb{Z}$. Thus Γ is a subgroup of

$$\Gamma' = \left\{ \begin{pmatrix} \sigma_1\lambda^{k_1} & 0 & \dots & 0 \\ 0 & \sigma_2\lambda^{k_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n\lambda^{k_n} \end{pmatrix} : \sigma_1, \dots, \sigma_n \in \mathbb{U}_m, k_1, \dots, k_n \in \mathbb{Z} \right\}.$$

Note Γ' is $\bar{\mathbb{R}}_\lambda$-definable. Abusing notation we let $\mathfrak{u} : (\mathbb{C}^\times)^n \to \mathbb{U}^n$ and we let $|\ | : (\mathbb{C}^\times)^n \to (\mathbb{R}_{>0})^n$ be given by

$$\mathfrak{u}(z_1, \dots, z_n) = (\mathfrak{u}(z_1), \dots, \mathfrak{u}(z_n)) \quad \text{and} \quad |(z_1, \dots, z_n)| = (|z_1|, \dots, |z_n|).$$

Then the map $(\mathbb{C}^\times)^n \to \mathbb{U}^n \times (\mathbb{R}_{>0})^n$ given by $\bar{z} \mapsto (\mathfrak{u}(\bar{z}), |\bar{z}|)$ restricts to a $\bar{\mathbb{R}}_\lambda$-definable isomorphism between Γ' and $\mathbb{U}_m^n \times (\lambda^{\mathbb{Z}})^n$. Lemma 6.3 below implies any subgroup of $\mathbb{U}_m^n \times (\lambda^{\mathbb{Z}})^n$ is $\bar{\mathbb{R}}_\lambda$-definable. □

We consider $(\mathbb{Z}/m\mathbb{Z}, +)$ to be a group with underlying set $\{0, \dots, m-1\}$ in the usual way so that $(\mathbb{Z}/m\mathbb{Z}, +)$ is a $(\mathbb{Z}, +)$-definable group. Lemma 6.3 is folklore. We include a proof for the sake of completeness.

Lemma 6.3. *Every subgroup H of $(\mathbb{Z}/m\mathbb{Z})^l \times \mathbb{Z}^n$ for $l \geq 0$ is $(\mathbb{Z}, +)$-definable.*

Proof. We first reduce to the case $l = 0$. The quotient map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is $(\mathbb{Z}, +)$-definable, it follows that the coordinate-wise quotient $\mathbb{Z}^l \times \mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^l \times \mathbb{Z}^n$ is $(\mathbb{Z}, +)$-definable. It suffices to show the preimage of H in $\mathbb{Z}^{l+n}$ is $(\mathbb{Z}, +)$-definable.

Suppose H is a subgroup of $\mathbb{Z}^n$. Then H is finitely generated with generators $\beta_1, \dots, \beta_k$ where $\beta_i = (b^i_1, \dots, b^i_n)$ for all $1 \leq i \leq k$. Then

$$H = \left\{ \sum_{i=1}^{k} c_i \beta_i : c_1, \dots, c_k \in \mathbb{Z} \right\}$$

$$= \left\{ \left(\sum_{i=1}^{k} c_i b^i_1, \dots, \sum_{i=1}^{k} c_i b^i_n \right) : c_1, \dots, c_n \in \mathbb{Z} \right\}.$$

Thus H is $(\mathbb{Z}, +)$-definable. □

7. Countable $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable groups

Fix $\lambda \in \mathbb{R}_{>0}$ and an o-minimal $\mathcal{R}$ with field of exponents $\mathbb{Q}$. Since $(\mathcal{R}, \lambda^{\mathbb{Z}})$ does not define $\mathbb{Z}$ by [18, Theorem 3.4.2], Theorem A implies every $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$ is virtually abelian. We extend this result to all countable interpretable groups.

Proposition 7.1. *Every countable $(\mathcal{R}, \lambda^{\mathbb{Z}})$-interpretable group is virtually abelian.*

Proposition 7.1 follows directly from several previous results. Every d-minimal expansion of $\bar{\mathbb{R}}$ admits definable selection by Miller [19]. Therefore an $(\mathcal{R}, \lambda^{\mathbb{Z}})$-interpretable group is isomorphic to an $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable group. We now recall two results of Tychonievich. The first is a special case of [27, 4.1.10].

Fact 7.2. *If $X \subseteq \mathbb{R}^k$ is $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable and countable, then there is an $\bar{\mathbb{R}}_\lambda$-definable surjection $f : (\lambda^{\mathbb{Z}})^m \to X$ for some m.*

Fact 7.3 is a minor rewording of [27, 4.1.2].

Fact 7.3. *Every $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable subset of $(\lambda^{\mathbb{Z}})^m$ is $(\lambda^{\mathbb{Z}}, <, \cdot)$-definable.*

Facts 7.2 and 7.3 together imply that every countable $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable group is isomorphic to a $(\mathbb{Z}, <, +)$-definable group. Now apply the following result of Onshuus and Vicaria [23] to complete the proof of Proposition 7.1.

Fact 7.4. *Every $(\mathbb{Z}, <, +)$-definable group is virtually abelian.*

References

[1] O. Belegradek and B. Zilber. The model theory of the field of reals with a subgroup of the unit circle. *J. Lond. Math. Soc. (2)*, 78(3):563–579, 2008.

[2] E. Caulfield. *Classifying expansions of the real field by complex subgroups*. ProQuest LLC, Ann Arbor, MI, 2018. Thesis (Ph.D.)–UIUC.

[3] C. Chevalley. *Theory of Lie Groups. I.* Princeton Mathematical Series, vol. 8. Princeton University Press, Princeton, N. J., 1946.

[4] L. v. d. Dries. The field of reals with a predicate for the powers of two. *Manuscripta Math.*, 54(1-2):187–195, 1985.

[5] L. v. d. Dries and A. Günaydın. The fields of real and complex numbers with a small multiplicative group. *Proc. London Math. Soc. (3)*, 93(1):43–81, 2006.

[6] R. Grigorchuk and I. Pak. Groups of intermediate growth: an introduction. *Enseign. Math. (2)*, 54(3-4):251–272, 2008.

[7] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[8] A. Gunaydin. *Model theory of fields with multiplicative groups*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.

[9] A. Günaydin and P. Hieronymi. Dependent pairs. *J. Symbolic Logic*, 76(2):377–390, 2011.

[10] J. Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.
[11] P. Hieronymi. Defining the set of integers in expansions of the real field by a closed discrete set. *Proc. Amer. Math. Soc.*, 138(6):2163–2168, 2010.
[12] P. Hieronymi. Expansions of subfields of the real field by a discrete set. *Fund. Math.*, 215(2):167–175, 2011.
[13] P. Hieronymi. The real field with an irrational power function and a dense multiplicative subgroup. *J. Lond. Math. Soc. (2)*, 83(1):153–167, 2011.
[14] P. Hieronymi and C. Miller. Metric dimensions and tameness in expansions of the real field. *Preprint*, 2015.
[15] P. Hieronymi and T. Nell. Distal and non-distal pairs. *J. Symb. Log.*, 82(1):375–383, 2017.
[16] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
[17] A. I. Mal' cev. On some classes of infinite soluble groups. *Mat. Sbornik N.S.*, 28(70):567–588, 1951.
[18] C. Miller. Tameness in expansions of the real field. In *Logic Colloquium '01*, volume 20 of *Lect. Notes Log.*, pages 281–316. Assoc. Symbol. Logic, Urbana, IL, 2005.
[19] C. Miller. Definable choice in d-minimal expansions of ordered groups. *https://people.math.osu.edu/miller.1987/eidmin.pdf*, 2006. Unpublished note.
[20] C. Miller and P. Speissegger. Expansions of the real line by open sets: o-minimality and open cores. *Fund. Math.*, 162(3):193–208, 1999.
[21] J. Milnor. Growth of finitely generated solvable groups. *J. Differential Geometry*, 2:447–449, 1968.
[22] G. A. Noskov. The elementary theory of a finitely generated almost solvable group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(3):498–517, 1983.
[23] A. Onshuus and M. Vicaria. Definable groups in models of presburger arthimetic and g^{00}. *preprint*, 2017.
[24] Y. Peterzil, A. Pillay, and S. Starchenko. Linear groups definable in o-minimal structures. *J. Algebra*, 247(1):1–23, 2002.
[25] J. Robinson. Definability and decision problems in arithmetic. *J. Symbolic Logic*, 14:98–114, 1949.
[26] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.
[27] M. A. Tychonievich. *Tameness results for expansions of the real field by groups*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The Ohio State University.
[28] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.
[29] J. A. Wolf. Growth of finitely generated solvable groups and curvature of Riemanniann manifolds. *J. Differential Geometry*, 2:421–446, 1968.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801
E-mail address: phierony@illinois.edu
URL: http://www.math.illinois.edu/~phierony

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801
E-mail address: erikw@illinois.edu
URL: http://www.math.illinois.edu/~erikw

School of Social Work, University of Illinois at Urbana-Champaign, 1010 West Nevada Street, Urbana, IL 61801
E-mail address: samxu@illinois.edu