COARSE DIMENSION AND DEFINABLE SETS IN EXPANSIONS OF THE ORDERED REAL VECTOR SPACE

ERIK WALSBERG

ABSTRACT. Suppose $E \subseteq \mathbb{R}$ is nowhere dense. If $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, E)$ does not define every compact subset of every $\mathbb{R}^n$ then for every $s > 0$ we have

$$|\{k \in \mathbb{Z}, -m \leq k \leq m-1 : [k, k+1] \cap E \neq \emptyset\}| < m^s$$

for all sufficiently large $m \in \mathbb{N}$.

1. Introduction

Let $\mathbb{R}_{\text{Vec}}$ be the ordered vector space $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$ of real numbers. For any subset E of $\mathbb{R}$ let $(\mathbb{R}_{\text{Vec}}, E)$ be the expansion of $\mathbb{R}_{\text{Vec}}$ by a unary predicate defining E. Hieronymi and Tychonievich [5] showed that $(\mathbb{R}_{\text{Vec}}, \mathbb{Z})$ defines all compact subsets of all $\mathbb{R}^n$ (in contrast to $(\mathbb{R}, <, +, \mathbb{Z})$, which admits quantifier elimination [7, 8]). We extend this result. The present paper is part of the broader study of the metric geometry of definable sets in first order structures expanding $(\mathbb{R}, <, +)$, see [1, 4, 2].

We recall previous work on first order expansions of $\mathbb{R}_{\text{Vec}}$. We let $\text{Cl}(E)$ be the closure of $E \subseteq \mathbb{R}$ and $\text{Bd}(E)$ be the boundary of E. Recall that the boundary of a subset of $\mathbb{R}$ is always closed. Fact 1.1 below follows by [2, Theorem 7.3, Corollary 7.5]. The implication (3) $\Rightarrow$ (2) is a corollary of a result of Friedman and Miller [3]. The implication (1) $\Rightarrow$ (3) is a corollary of a fundamental result of Hieronymi and Tychonievich [5].

Fact 1.1. *Suppose that $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. Then the following are equivalent:*

(1) $(\mathbb{R}_{Vec}, E)$ does not define every compact subset of every $\mathbb{R}^n$,
(2) Every $(\mathbb{R}_{Vec}, E)$-definable subset of $\mathbb{R}$ either has interior or is nowhere dense,
(3) $T(\text{Bd}(E)^n)$ is nowhere dense for every linear $T : \mathbb{R}^n \to \mathbb{R}$.

Note that $\text{Bd}(E)$ is nowhere dense as E is not dense and co-dense in any open interval. If E is bounded then (3) above is equivalent to a natural geometric condition on E. The equivalence, observed in [2, Theorem 7.3], is an easy consequence of the famous Marstrand projection theorem (see [6, Chapter 9]) and the classical theorem of Steinhaus that $X - X$ has interior whenever $X \subseteq \mathbb{R}^n$ has positive n-dimensional Lebesgue measure.

Fact 1.2. *Suppose $F \subseteq \mathbb{R}$ is bounded and nowhere dense. Then $T(F^n)$ is nowhere dense for every linear $T : \mathbb{R}^n \to \mathbb{R}$ if and only if $\text{Cl}(F)^n$ has Hausdorff dimension zero for all n.*

Fact 1.2 does not hold for unbounded subsets of $\mathbb{R}$. The set of integers, like any countable set, has Hausdorff dimension zero, and $T(\mathbb{Z}^2)$ is dense in $\mathbb{R}$ for any linear $T : \mathbb{R}^2 \to \mathbb{R}$ of the form $T(x, y) = \alpha x + \beta y$ with $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{R} \setminus \mathbb{Q}$. In the present note we give a proof of the following:

Theorem 1.3. *Let $F \subseteq \mathbb{R}$ be nowhere dense. If there is a real number $s > 0$ such that*

$$|\{k \in \mathbb{Z}, -m \leq k \leq m-1 : [k, k+1] \cap F \neq \emptyset\}| \geq m^s$$

for arbitrarily large $m \in \mathbb{N}$, then $T(F^n)$ is somewhere dense in $\mathbb{R}$ for some linear $T : \mathbb{R}^n \to \mathbb{R}$.

2. Metric Notions

We recall some notions from metric geometry and describe the coarse Minksowski dimension.

Let $X \subseteq \mathbb{R}^n$ be bounded. Given $\delta > 0$ we let $M(\delta, X)$ be the minimum number of open δ-balls required to cover X. Equivalently, $M(\delta, X)$ is the minimal cardinality of a subset S of X such that every $x \in X$

Date: March 4, 2019.

lies within distance δ of some element of S. We also define

$$N(X) := \left| \left\{ (k_1, \dots, k_n) \in \mathbb{Z}^n : X \cap \prod_{i=1}^{n} [k_i, k_i + 1] \neq \emptyset \right\} \right|.$$

We recall several facts about these well-known invariants, all of which are easy to see. One can find more information about such invariants in [9, Chapter 2] and many other places.

Fact 2.1. *There is a constant $C > 0$ depending only on n such that*

$$M(\delta', X) \leq M(\delta, X) \leq C \left(\frac{\delta'}{\delta} \right)^n M(\delta', X)$$

for all $0 < \delta < \delta'$ and bounded $X \subseteq \mathbb{R}^n$. There is a constant $K > 0$ depending only on n such that

$$K^{-1} M(1, X) \leq N(X) \leq K M(1, X)$$

for all bounded $X \subseteq \mathbb{R}^n$. Finally, there is a constant $L > 0$ depending only on n such that

$$L^{-1} M(\delta, X) M(\delta, Y) \leq M(\delta, X \times Y) \leq L M(\delta, X) M(\delta, Y)$$

for all bounded $X, Y \subseteq \mathbb{R}^n$, so in particular

$$L^{-1} M(\delta, X)^2 \leq M(\delta, X^2) \leq L M(\delta, X)^2$$

for any bounded $X \subseteq \mathbb{R}^n$.

The δ-**entropy** of X is $H_\delta(X) := \log M(\delta, X)$. Let $B_n(p, r)$ be the open ball in $\mathbb{R}^n$ with center p and radius $r > 0$ and let $B_n(r) = B_n(0, r)$. Given a possibly unbounded $Z \subseteq \mathbb{R}^n$ we define the **coarse Minkowski dimension** of Z to be

$$\dim_{\mathrm{CM}}(X) := \limsup_{r \to \infty} \frac{H_1(B_n(r) \cap Z)}{\log(r)}.$$

Let's make a few observations. An application of the first claim of Fact 2.1 above shows that replacing 1 with any $\delta > 0$ does not change the coarse Minkowski dimension. It is easy to see that the coarse Minkowski dimension of a subset of $\mathbb{R}^n$ cannot exceed n and that the coarse Minkowski dimension of a bounded set is zero. A simple computation shows that $\dim_{\mathrm{CM}}(X)$ is the infimum of the set of $s > 0$ such that $M(1, B_n(r) \cap X) < r^s$ for all sufficiently large $r > 0$. It follows from the second claim of Fact 2.1 that $\dim_{\mathrm{CM}}(X) = 0$ if and only if for every $s > 0$ we have $N(B_n(r) \cap X) < r^s$ for all sufficiently large $r > 0$.

The proof of the fact below is a straightforward computation which is essentially the same as the proof of the analogous fact for Minkowski dimension. We leave the proof to the reader. We obtain an inequality in the first claim and an equality in the second because $\limsup_{n \to \infty}(a_n + b_n)$ may be strictly less then $\limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$, but $\limsup_{n \to \infty}(2a_n)$ is always equal to $2 \limsup_{n \to \infty} a_n$.

Fact 2.2. *For any $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ we have*

$$\dim_{CM}(X \times Y) \leq \dim_{CM}(X) + \dim_{CM}(Y)$$

and

$$\dim_{CM}(X^2) = 2 \dim_{CM}(X).$$

Suppose that $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ and $\lambda, \delta > 0$. A map $f : X \to Y$ is a (λ, δ)-quasi-isometry if

$$\frac{1}{\lambda} \|x - x'\| - \delta \leq \|f(x) - f(x')\| \leq \lambda \|x - x'\| + \delta \quad \text{for all} \quad x, x' \in X,$$

and if for every $y \in Y$ we have $\|f(x) - y\| < \delta$ for some $x \in X$. We say that $f : X \to Y$ is a quasi-isometry if it is a (λ, δ)-quasi-isometry for some $\lambda, \delta > 0$. It is well-known and easy to see that if there is a quasi-isometry $X \to Y$ then there is also a quasi-isometry $Y \to X$. A map $f : X \to \mathbb{R}^n$ is a quasi-isometric embedding if it restricts to a quasi-isometry $X \to f(X)$.

There should be a more general version of Lemma 2.3. To avoid technicalities we only prove what we need below.

Lemma 2.3. *Suppose $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m, 0 \in X, 0 \in Y$, and $f : X \to Y$ is a quasi-isometry such that $f(0) = 0$. Then X and Y have the same coarse Minkowski dimension.*

Proof. We show that $\dim_{\mathrm{CM}}(Y) \le \dim_{\mathrm{CM}}(X)$. As there is a quasi-isometry $Y \to X$ which also maps 0 to 0 the same argument yields the other inequality. Suppose that f is a (λ, δ)-quasi-isometry.

Fix $r > 0$. Let $X_r = B_n(0, r) \cap X$ and $Y_r = B_m(0, r) \cap Y$. Let $\{B_n(p_i, 1)\}_{i=1}^k$ be a minimal covering of X_r by balls with radius 1. Then $\{f(B_n(p_i, 1))\}_{i=1}^k$ covers $f(X_r)$. Let $q_i = f(p_i)$ for all i. As f is a (λ, δ)-quasi-isometry we see that $f(B_n(p_i, 1))$ is contained in $B_m(q_i, \lambda + \delta)$ for all i. Thus $\{B_m(q_i, \lambda + \delta)\}_{i=1}^k$ covers $f(X_r)$.

We now show that every point in $Y_{r\lambda^{-1}-2\delta}$ lies within distance δ of $f(X_r)$. Fix $y \in Y_{r\lambda^{-1}-2\delta}$. As f is a (λ, δ)-quasi-isometry there is an $x \in X$ such that $\|f(x) - y\| < \delta$. Suppose $\|x\| > r$. Then as $f(0) = 0$ we have

$$\|f(x)\| \ge \frac{1}{\lambda}\|x\| - \delta > r\lambda^{-1} - \delta.$$

As $\|f(x) - y\| < \delta$ the triangle inequality yields $\|y\| > r\lambda^{-1} - 2\delta$. Contradiction.

Combining the previous paragraphs we see that $\{B_m(q_i, \lambda + 2\delta)\}_{i=1}^k$ covers $Y_{r\lambda^{-1}-2\delta}$. Thus

$$M(\lambda + 2\delta, Y_{r\lambda^{-1}-2\delta}) \le M(1, X_r) \quad \text{for all} \quad r > 0.$$

Applying the first claim of Fact 2.1 we obtain a constant $L > 0$ depending only on m such that

$$LM(1, Y_{r\lambda^{-1}-2\delta}) \le M(\lambda + 2\delta, Y_{r\lambda^{-1}-2\delta})$$

hence

$$LM(1, Y_{r\lambda^{-1}-2\delta}) \le M(1, X_r).$$

Taking log's, dividing through by $\log(r)$, taking the limit as $r \to \infty$, and applying the fact that

$$\lim_{r\to\infty} \frac{\log(r\lambda^{-1} - 2\delta)}{\log(r)} = 1$$

shows that $\dim_{\mathrm{CM}}(Y) \le \dim_{\mathrm{CM}}(X)$. □

3. The proof

We prove two Euclidean lemmas. Let $\mathbb{S}$ be the unit circle in $\mathbb{R}^2$. Given $u \in \mathbb{S}$ we let $T_u : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection parallel to u, i.e., T_u is the orthogonal projection such that $T_u(x) = T_u(y)$ if and only if $x - y = tu$ for some $t \in \mathbb{R}$. For our purposes a **double wedge** around $u \in \mathbb{S}$ is a subset of $\mathbb{R}^2$ of the form

$$C^u_{s,\varepsilon} := \{tv : t \in \mathbb{R}, |t| > s, v \in \mathbb{S}, \|v - u\| < \varepsilon\}$$

for some $s, \varepsilon > 0$. Lemma 3.1 is a quasi-isometric version of a well known projection trick. The biLipschitz version of this trick is used in [1].

Lemma 3.1. *Let F be a nonempty subset of $\mathbb{R}^2$. If $F - F := \{x - y : x, y \in F\}$ is disjoint from some double wedge around $u \in \mathbb{S}$ then T_u restricts to a quasi-isometric embedding $F \to \mathbb{R}$.*

Proof. Suppose that $F - F$ is disjoint from $C^u_{s,\varepsilon}$. As T_u is an othogonal projection we have $\|T_u(x) - T_u(x')\| \le \|x - x'\|$ for all $x, x' \in \mathbb{R}^2$, so it suffices to obtain a lower bound on $\|T_u(X) - T_u(x')\|$ of the appropriate form.

After making a change of coordinates if necessary we suppose $u = (0, 1)$ so that $T_u(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Then we have

$$C^u_{s,\varepsilon} = \{(x, y) \in \mathbb{R}^2 : |y| > \lambda|x|, \|(x, y)\| > s\}$$

for some $\lambda > 0$ depending only on ε. Thus, if $(x, y) \in F - F$ then either $\|(x, y)\| < s$ or $|y| \le \lambda|x|$. That is, for all $(x, y), (x', y') \in F$ we either have

$$\|(x, y) - (x', y')\| < s \quad \text{or} \quad |y - y'| \le \lambda|x - x'|.$$

In the latter case we have

$$\|(x, y) - (x', y')\| \le |x - x'| + |y - y'| \le (1 + \lambda)|x - x'|$$

hence

$$\frac{1}{1+\lambda}\|(x, y) - (x', y')\| \le |x - x'|.$$

In the first case we have

$$\|(x, y) - (x', y')\| - s < |x - x'|.$$

So in either case

$$\frac{1}{1+\lambda}\|(x, x') - (y, y')\| - s \le |x - x'|.$$

Thus T restricts to a quasi-isometric embedding $F \to \mathbb{R}^2$. □

We let $\mathbb{H}$ be the upper half plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$ and let $\mathbb{S}^+ = \mathbb{S} \cap \mathbb{H}$. A wedge in $\mathbb{H}$ is a set of the form

$$C^{u,+}_{s,\varepsilon} := \{tv : t \in \mathbb{R}, t > s, v \in \mathbb{S}, \|v - u\| < \varepsilon\}$$

such that $C^{u,+}_{s,\varepsilon} \subseteq \mathbb{H}$.

Lemma 3.2. *Suppose $F \subseteq \mathbb{H}$ intersects every wedge in $\mathbb{H}$. Then $T_u(F)$ is dense in $\mathbb{R}$ for some $u \in \mathbb{S}^+$.*

The reader may find that drawing a few pictures greatly assists in comprehending the proof of Lemma 3.2. We let $p = (-1,0)$ and $o = (0,0)$. Note that if $z \in \mathbb{H}$, q is a positive real number, and $u \in \mathbb{S}^+$, then $T_u(z) = q$ if and only if $\angle pou = \angle pqz$.

Proof. We show that the set of $u \in \mathbb{S}^+$ such that $T_u(F)$ is dense in $\mathbb{R}$ is comeager in $\mathbb{S}^+$. It suffices to show that

$$\{u \in \mathbb{S}^+ : T_u(F) \cap I \neq \emptyset\}$$

is open and dense in $\mathbb{S}^+$ for every open interval I with rational endpoints. Fix an open interval $I = (q_1, q_2)$ with rational endpoints. We suppose that $q_1, q_2 > 0$ for the sake of simplicity, the more general case follows by trivial altercations of our argument. The map $T : \mathbb{S}^+ \times \mathbb{R}^2 \to \mathbb{R}$ given by $T(u,x) = T_u(x)$ is continuous. Thus if $T_u(x) \in I$ then $T_v(x) \in I$ for all $v \in \mathbb{S}^+$ sufficiently close to u. It follows that the set of u such that $T_u(F) \cap I \neq \emptyset$ is open in $\mathbb{S}^+$.

We show that the set of $w \in \mathbb{S}^+$ such that $T_w(F) \cap I \neq \emptyset$ is dense in $\mathbb{S}^+$. Fix $u, v \in \mathbb{S}^+$ such that $\angle pou < \angle pov$ and let J be the set of $w \in \mathbb{S}^+$ such that $\angle pou < \angle pow < \angle pov$. We show there is a $w \in J$ such that $T_w(F) \cap I \neq \emptyset$. Let $r_1, r_2 \in \mathbb{H}$ be such that $\angle pq_1r_1 = \angle pou$ and $\angle pq_2r_2 = \angle pov$. Let D be the set of points in $\mathbb{H}$ that lie in between the rays $\overrightarrow{q_1r_1}$ and $\overrightarrow{q_2r_2}$. It is easy to see that

$$D = \bigcup_{q \in I} \{r \in \mathbb{H} : \angle pou < \angle pqr < \angle pov\} = \bigcup_{q \in I} \bigcup_{w \in J} T_w^{-1}(\{q\}) = \bigcup_{w \in J} T_w^{-1}(I).$$

It therefore suffices to show that D intersects F. Let $z_1, z_2 \in \mathbb{S}^+$ be such that

$$\angle pou < \angle poz_1 < \angle poz_2 < \angle pov.$$

As $\angle pq_1r_1 < \angle poz_1 < \angle poz_2 < \angle pq_2r_2$, we see that every element of $\overrightarrow{oz_1}$ or $\overrightarrow{oz_2}$ sufficiently far from the origin lies in D. It follows that there is a $t > 0$ such that

$$W := \{z \in \mathbb{H} : \|z\| \geq t, \angle poz_1 < \angle poz < \angle poz_2\} \subseteq D.$$

This W is a wedge in $\mathbb{H}$ and so contains an element of F. Thus D contains an element of F. □

Lemma 3.3. *Suppose $E \subseteq \mathbb{R}$. Then one of the following holds:*

1. *T_u restricts to a quasi-isometric embedding $E^2 \to \mathbb{R}$ for some $u \in \mathbb{S}$,*
2. *$S(E^4)$ is dense in $\mathbb{R}$ for some linear $S : \mathbb{R}^4 \to \mathbb{R}$.*

Proof. Consider $E^2 - E^2 \subseteq \mathbb{R}^2$. If $E^2 - E^2$ is disjoint from a double wedge in $\mathbb{R}^2$ then Lemma 3.2 shows that some T_u quasi-isometrically embedds E^2 into $\mathbb{R}$.

Suppose $E^2 - E^2$ intersects every double wedge in $\mathbb{R}^2$. Note that if $(x,y) \in E^2 - E^2$ then $(-x,-y)$ is also an element of $E^2 - E^2$. It is easy to see that this implies that $E^2 - E^2$ intersects every wedge in $\mathbb{H}$. Applying Lemma 3.3 we fix a $u \in \mathbb{S}$ such that $T_u(E^2 - E^2)$ is dense in $\mathbb{R}$. Let $S : \mathbb{R}^4 \to \mathbb{R}$ be the linear function given by

$$S(x,y,x',y') = T_u(x - x', y - y') \quad \text{for all} \quad x,y,x',y' \in \mathbb{R}.$$

Then $S(E^4)$ is dense in $\mathbb{R}$. □

Proposition 3.4. *If $E \subseteq \mathbb{R}$ has positive coarse Minkowski dimension then $T(E^n)$ is dense in $\mathbb{R}$ for some linear $T : \mathbb{R}^n \to \mathbb{R}$.*

Proof. We suppose that E has positive coarse Minkowski dimension and that $T : E^n \to \mathbb{R}$ is not dense in $\mathbb{R}$ for any linear $T : \mathbb{R}^n \to \mathbb{R}$ towards a contradiction. We may also suppose that $0 \in E$. Let $\mathcal{S}$ be the collection of sets of the form $T(E^n)$ for linear $T : \mathbb{R}^n \to \mathbb{R}$. It is easy to see that if $F \in \mathcal{S}$ and $T : \mathbb{R}^n \to \mathbb{R}$ is linear then $T(F^n)$ is also in $\mathcal{S}$. We let s be the supremum of the coarse Minkowski dimensions of members of $\mathcal{S}$. Every element of $\mathcal{S}$ has coarse Minkowski dimension ≤ 1, so s exists and $0 < s \leq 1$. Let $F \in \mathcal{S}$ be such that $2\dim_{\mathrm{CM}}(F) > s$. An application of Lemma 3.3 yields a linear $T : \mathbb{R}^2 \to \mathbb{R}$ which restricts to a quasi-isometry $F^2 \to \mathbb{R}$. Lemma 2.3 and Fact 2.2 shows that

$$\dim_{\mathrm{CM}} T(F^2) = \dim_{\mathrm{CM}}(F^2) = 2\dim_{\mathrm{CM}}(F) > s.$$

But $T(F^2) \in \mathcal{S}$, contradiction. □

References

1. Antongiulio Fornasiero, Philipp Hieronymi, and Chris Miller, *A dichotomy for expansions of the real field*, Proc. Amer. Math. Soc. **141** (2013), no. 2, 697–698. MR 2996974
2. Antongiulio Fornasiero, Philipp Hieronymi, and Erik Walsberg, *How to avoid a compact set*, Adv. Math. **317** (2017), 758–785.
3. Harvey Friedman and Chris Miller, *Expansions of o-minimal structures by sparse sets*, Fund. Math. **167** (2001), no. 1, 55–64. MR 1816817
4. Philipp Hieronymi and Chris Miller, *Metric dimensions and tameness in expansions of the real field*, Preprint (2015).
5. Philipp Hieronymi and Michael Tychonievich, *Interpreting the projective hierarchy in expansions of the real line*, Proc. Amer. Math. Soc. **142** (2014), no. 9, 3259–3267. MR 3223381
6. Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability. MR 1333890
7. Chris Miller, *Expansions of dense linear orders with the intermediate value property*, J. Symbolic Logic **66** (2001), no. 4, 1783–1790. MR 1877021 (2003j:03044)
8. Volker Weispfenning, *Mixed real-integer linear quantifier elimination*, Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC), ACM, New York, 1999, pp. 129–136 (electronic). MR 1802076 (2002b:03071)
9. Yosef Yomdin and Georges Comte, *Tame geometry with application in smooth analysis*, Lecture Notes in Mathematics, vol. 1834, Springer-Verlag, Berlin, 2004. MR 2041428

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

E-mail address: erikw@illinois.edu

URL: http://www.math.illinois.edu/~erikw