MARKER-STEINHORN VIA DEFINABLE LINEAR ORDERS

Abstract. We give a short proof of the Marker-Steinhorn theorem for o-minimal expansions of ordered groups. The key tool is Ramakrishnan’s classification of definable linear orders in such structures.

1. INTRODUCTION

Let \( M = (M, \leq, \ldots) \) be an o-minimal expansion of a dense linear order without endpoints, possibly with additional structure, in the language \( L \). A type \( p(x) \) over \( M \) is definable if for every \( L \)-formula \( \delta = \delta(x, y) \) in the (object) variables \( x = (x_1, \ldots, x_m) \) and (parameter) variables \( y = (y_1, \ldots, y_n) \) there is a defining formula for the restriction \( p \restriction \delta \), i.e. a formula \( \phi(y) \), possibly with parameters from \( M \), such that \( \delta(x, b) \in p \Leftrightarrow M \models \phi(b) \), for all \( b \in M^n \).

A set \( C \subseteq M \) is a cut in \( M \) if whenever \( c \in C \), then \(( -\infty, c) := \{ a \in M : a < c \}\) is contained in \( C \). Let \( \delta(x, y) \) be the formula \( x > y \) (in the language of \( M \)). It is well-known that cuts in \( M \) correspond in a one-to-one way to complete \( \delta \)-types over \( M \), where to the cut \( C \) in \( M \) we associate the complete \( \delta \)-type \( p_C(x) := \{ \delta(x, b) : b \in C \} \cup \{ \neg \delta(x, b) : b \in M \setminus C \} \).

The \( \delta \)-type \( p_C \) is definable if and only if the cut \( C \) in \( M \) is definable. If \( C \) is of the form \(( -\infty, c) := \{ a \in M : a \leq c \}\) or \(( -\infty, c) \subseteq (M \cup \{ -\infty \}, \leq) \) or \(( -\infty, c) \subseteq (M \cup \{ +\infty \}, \leq) \), then \( C \) is definable. Such cuts are said to be rational. It follows from o-minimality that all definable cuts are rational. If \( (M, \leq) = (\mathbb{R}, \leq) \) then all cuts in \( M \) are rational. This can be used to define the standard part map for elementary extensions. That is, if \( (M, \leq) = (\mathbb{R}, \leq) \) and \( M \leq M^* = (M^*, \leq, \ldots) \) then we define the standard part map \( b \mapsto \sup \{ a \in M : a \leq b \} : M^* \cup \{ +\infty \} \to M \cup \{ +\infty \} \),

where we declare \( \sup \emptyset := -\infty \) and \( \sup M := +\infty \). To generalize this, we say an elementary extension \( M \leq M^* \) is tame if for every \( a \in M^* \) the cut \( \{ b \in M : b \leq a \} \) is rational. (Thus if \( (M, \leq) = (\mathbb{R}, \leq) \) then every elementary extension of \( M \) is tame.) We can then define the standard part map in the same way.

It follows by o-minimality that every 1-type over \( M \) is determined by its restriction to \( \delta \), so a 1-type over \( M \) is definable exactly when the associated cut in \( M \) is rational. It trivially follows that \( M \leq M^* \) is tame if and only if for every \( a \in M^* \), the type \( \text{tp}(a|M) \) is definable. Marker and Steinhorn [3] generalized this to show that \( M \leq M^* \) is tame then for every \( a \in (M^*)^m \), the type \( \text{tp}(a|M) \) is definable. In particular if \( (M, \leq) = (\mathbb{R}, \leq) \) then every type over \( \mathbb{R} \) is definable. See [9] for geometric applications of this useful result. The original proof of Marker and Steinhorn uses a complicated inductive argument. Tressl [7] proved the Marker-Steinhorn for o-minimal expansions of real closed fields with a short and clever argument using valuation theory and co-heirs that gives little idea as to the form of the defining
formula of a type. Chernikov and Simon gave a proof using NIP-theoretic machinery [2]. We give a more constructive proof of the Marker-Steinhorn Theorem for \( o \)-minimal expansions of ordered groups. The crucial idea is to reduce the analysis of \( n \)-types to an analysis of cuts in definable linear orders. Our main tool is the following theorem of Ramakrishnan [5], which is closely related to earlier work of Onshuus-Steinhorn [4]. Let \( \preceq_{lex} \) be the lexicographic order on \( M^k \).

**Theorem 1.1.** Suppose \( M \) expands an ordered group. Then every definable linear order is definably isomorphic to a definable subset of some \( M^k \) equipped with the induced lexicographic order.

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2. **Conventions**

Throughout, \( M \) is an \( o \)-minimal expansion of an ordered abelian group, and \( M \preceq M^* = (M^*, \ldots) \) is a tame elementary extension. Unless said otherwise, “definable” means “definable, possibly with parameters,” and the adjective “definable” applied to subsets of \( M^m \) or maps \( A \to M^n, A \subseteq M^m \), will mean “definable in \( M \).” The basic facts about \( o \)-minimal structures that we use may be found in [8]. We let \( m, n, k, l \) range over natural numbers. Given sets \( A, B, C \subseteq A \times B \), and \( a \in A \) we let

\[ C_a := \{ b \in B : (a, b) \in C \}. \]

If \( A \subseteq M^m \) is a definable set, then \( A^* \) denotes the subset of \((M^*)^m \) defined in \( M^* \) by the same formula (since \( M \preceq M^* \), this does not depend on the choice of a defining formula). Similarly, if \( f : A \to M^n \) is a definable map, then \( f^* : A^* \to (M^*)^n \) denotes the map whose graph is defined in \( M^* \) by the same formula as the graph of \( f \). If \( A \subseteq M^k \) is definable then \( \dim(A) \) is the usual \( o \)-minimal dimension of \( A \). Given a bounded definable \( A \subseteq M^* \) we let \( \mu(A) \) be the sum of the lengths of the components of \( A \). If \( A \subseteq (M^*)^m \times M^* \) is such that every \( A_x \) is bounded then there is a definable \( f : (M^*)^n \to M^* \) such that \( f(x) = \mu(A_x). \) We call \( \mu(A) \) the measure of \( A \). (Indeed, \( \mu \) is a finitely additive measure on the collection of bounded \( M^* \)-definable subsets of \( M^* \)).

3. **Cuts in Definable Linear Orders**

Throughout this section \( (P, \preceq_P) \) is a definable linear order and \( P \subseteq M^m \).

**Proposition 3.1.** If \( V \subseteq P^* \) is \( M^* \)-definable and \( W = V \cap P \) is a cut in \( P^* \), then \( W \) is definable.

The proof of this proposition is the most difficult part of this paper. The difficulty largely lies in the fact that \( V \) may not be a cut in \( P^* \). We need the following three lemmas for Proposition 3.1. The first is an easy base case of the Marker-Steinhorn theorem, which we leave to the reader.

**Lemma 3.2.** If \( A \subseteq M^* \) is \( M^* \)-definable then \( A \cap M \) is definable.

The second lemma follows easily from \( o \)-minimality, we leave the proof to the reader.
Lemma 3.3. Suppose $I \subseteq M$ is a bounded interval and $J \subseteq I^*$ is $\mathcal{M}^*$-definable. If $\mu(J) \geq \frac{1}{2} \mu(I^*)$ then $J \cap I$ is nonempty. If $I \subseteq J$ then $\mu(J) \geq \frac{1}{2} \mu(I^*)$.

Lemma 3.3 holds with any rational $0 < q < 1$ in place of $\frac{1}{2}$.

Lemma 3.4. Suppose $A \subseteq M^m$ is definable and $B$ is an $\mathcal{M}^*$-definable subset of $A^*$ such that $A_x$ is either contained in or disjoint from $B_x$ for all $x \in M^{m-1}$. Then there is an $\mathcal{M}^*$-definable $D \subseteq (\mathcal{M}^*)^{m-1}$ such that

$$D \cap M^{m-1} = \{ x \in M^{m-1} : A_x \subseteq B_x \}.$$ 

Proof. Let $\{C_1, \ldots, C_n\}$ be a cell decomposition of $A$. Note that $(C_i)_x$ is either contained in or disjoint from $B_x$ for all $x \in M^{m-1}$ and $1 \leq i \leq n$. If $D_i$ is an $\mathcal{M}^*$-definable subset of $M^{m-1}$ such that

$$D_i \cap M^{m-1} = \{ x \in M^{m-1} : (C_i)_x \subseteq B_x \}$$

for $1 \leq i \leq n$, then $D := D_1 \cap \ldots \cap D_n$ satisfies the conditions of the lemma. We therefore assume $A$ is a cell. We now consider four cases. The first case is when $A$ is the graph of a continuous definable $f : A' \to M$ on a cell $A' \subseteq M^{m-1}$. In this case we take $D$ to be the set of $x \in (A')^*$ such that $f(x) \in B_x$. The second case is when

$$A = \{(x,t) \in M^{m-1} \times M : x \in A', f(x) < t < g(x)\}$$

for continuous definable $f, g : A' \to M$ on a cell $A' \subseteq M^{m-1}$ such that $f(x) < g(x)$ for all $x \in A'$. It follows from Lemma 3.3 that $A_x \subseteq B_x$ implies

$$\mu(B_x) \geq \frac{1}{2} \mu(A_x^*) = \frac{1}{2} \mu(\gamma(x) - \gamma^*(x))$$

for all $x \in A'$.

Lemma 3.3 also shows that for all $x \in A'$, if $\mu(B_x) \geq \frac{1}{2} \mu(A_x^*)$ then $A_x$ and $B_x$ intersect. We therefore take $D$ to be the set of $x \in (A')^*$ such that $\mu(B_x) \geq \frac{1}{2} \mu(A_x^*)$. The third case is when

$$A = \{(x,y) \in M^{m-1} \times M : x \in A', y > f(x)\}$$

for continuous definable $f : A' \to M$ on a cell $A' \subseteq M^{m-1}$. If $B_x$ contains $A_x$ then $B_x$ contains $\{ y \in M : f(x) < y < f(x) + 1 \}$. Conversely if $B_x$ contains $\{ y \in M : f(x) < y < f(x) + 1 \}$ then $B_x$ intersects $A_x$ and hence contains $A_x$ by assumption. Thus $B_x$ contains $A_x$ if and only if it contains $\{ y \in M : f(x) < y < f(x) + 1 \}$. Reasoning as before we take $D$ to be the set of $x \in (A')^*$ such that

$$\mu(A_x \cap \{ y \in M^* : f(x) < y < f(x) + 1 \}) \geq \frac{1}{2}.$$ 

The fourth case is when

$$A = \{(x,y) \in M^{m-1} \times M : x \in A', y < g(x)\}$$

for continuous definable $g : A' \to M$ on a cell $A' \subseteq M^{m-1}$. This case follows in the same way as the third case. \qed

We now prove Proposition 3.1

Proof. Applying Theorem 1.1 let $P' \subseteq M^k$ be definable, $\leq_{lex}$ be the restriction of the lexicographic order on $M^k$ to $P'$, and suppose $i : (P, \leq_P) \to (P', \leq_{lex})$ is a definable isomorphism of linear orders. It suffices to show $i(W) = i^*(V) \cap P'$ is definable. We therefore suppose $\leq_P$ is the restriction of the lexicographic order on $M^m$ to $P$. We apply induction on $m$. If $m = 1$ then $W$ is definable by Lemma 3.2. Suppose $m \geq 2$, let $\pi : P \to M^{m-1}$ be the projection onto the
first $m - 1$ coordinates, and let $Q = \pi(P)$. Note $\pi$ is a monotone map $(P, \leq_P) \to (Q, \leq_{\text{lex}})$, it follows that $\pi(W)$ is a cut in $(Q, \leq_{\text{lex}})$. We consider two cases:

(1) $\pi(W)$ has a maximum $q$ in $(Q, \leq_{\text{lex}})$.
(2) $\pi(W)$ does not have a maximum in $(Q, \leq_{\text{lex}})$.

We first treat case (1). The assumption implies that $\pi^{-1}(q) \cap W$ is upwards cofinal in $W$, so $W$ is the downwards closure of $\pi^{-1}(q) \cap W$. Lemma 3.2 shows $\pi^{-1}(q) \cap W$ is definable, so $W$ is definable.

We now treat case (2). If $p \in W$ then as $\pi(p)$ is not the maximal element of $\pi(W)$ it follows that $\pi(p') >_{\text{lex}} \pi(p)$ for some $p' \in W$, which implies $p' > q$ for all $q \in \pi^{-1}(p)$, as $W$ is downwards closed we have $\pi^{-1}(p) \subseteq W$. Thus, for any $p \in Q$, if $W$ intersects $\pi^{-1}(p)$ then $W$ contains $\pi^{-1}(p)$. Note in particular that this implies $W = \pi^{-1}(\pi(W))$, so it suffices to show $\pi(W)$ is definable. Applying Lemma 3.4 we obtain an $M^*$-definable $D \subseteq M^{m-1}$ such that $D \cap Q = \pi(W)$. As $\pi(W)$ is a cut in $(Q, \leq_{\text{lex}})$ the inductive hypothesis implies $\pi(W)$ is definable. □

The proof of Proposition 3.1 may be simplified by applying a result of Shelah, see [6] or [1]. This result, which holds for any NIP structure, implies that if $D \subseteq (M^*)^k$ is $M^*$-definable and $\pi : (M^*)^k \to (M^*)^l$ is a coordinate projection then there is an $M^*$-definable $E \subseteq (M^*)^l$ such that $\pi(D \cap M^k) = E \cap M^l$. Applying this result allows us to avoid the use of Lemma 3.4 and directly apply the inductive assumption to $\pi(W)$.

4. Proof of Marker-Steinhorn

Fix $b = (b_1, \ldots, b_k) \in (M^*)^k$. The following theorem shows $\text{tp}(b|M)$ is definable.

**Theorem 4.1.** If $A \subseteq M^i \times M^k$ is definable then $\{a \in M^i : (a, b) \in A^*\}$ is also definable.

**Proof.** We apply induction on $k$. The base case $k = 1$ holds as all 1-types over $M$ realized in $M^*$ are definable. Suppose $k \geq 2$ and let $b' = (b_1, \ldots, b_{k-1})$. We declare $\text{dim}(b|M)$ to be the minimal dimension of a definable $B \subseteq M^k$ such that $b \in B^*$. We first consider the case $\text{dim}(b|M) < k$. Let $B \subseteq M^k$ be definable such that $b \in B^*$ and $\text{dim}(B) < k$. Let $\{C_1, \ldots, C_n\}$ be a cell decomposition of $B$, let $1 \leq i \leq n$ be such that $b \in (C_i)^*$. After replacing $B$ with $C_i$ if necessary we suppose $B$ is a cell. As $\text{dim}(B) < k$ we suppose, after permuting coordinates if necessary, that

$$B = \{(a, t) \in M^{k-1} \times M : a \in B', t = f(a)\}$$

for a cell $B' \subseteq M^{k-1}$ and a continuous definable $f : B' \to M$. Note $b_k = f^*(b')$.

Let $E$ be the set of $(a, c) \in M^i \times M^{k-1}$ such that $(a, c, f(c)) \in A$. Given $a \in M^i$, we have $(a, b) \in A^*$ if and only if $(a, b') \in E^*$. Applying the inductive hypothesis to $b'$ shows $\{a \in M^i : (a, b') \in E^*\}$ is definable. We therefore suppose $\text{dim}(b|M) = k$.

Suppose $\{C_1, \ldots, C_n\}$ is a cell decomposition of $A$. It suffices to show that $\{a \in M^i : (a, b) \in (C_i)^*\}$ is definable for $1 \leq i \leq n$. We therefore suppose $A$ is a cell. We suppose without loss of generality that $\{a \in M^i : (a, b) \in A^*\}$ is nonempty. As $\text{dim}(b|M) = k$ it follows that $\text{dim}(A_x) = k$ for some $x \in M^i$. As $A$ is a cell it follows that $\text{dim}(A_x) = k$ for all $x \in M^i$ such that $A_x \neq \emptyset$, so each $A_x$ is an open cell. Then one of the following holds:

- $A = \{(a, c, t) \in M^i \times M^{k-1} \times M : (a, c) \in A', f(a, c) < t < g(a, c)\}$,
- $A = \{(a, c, t) \in M^i \times M^{k-1} \times M : (a, c) \in A', t < g(a, c)\}$. 


\[ A = \{(a,c,t) \in M^I \times M^{k-1} \times M : (a,c) \in A', f(a,c) < t\}, \]
for a cell \( A' \subseteq M^I \times M^{k-1} \) and continuous definable \( f,g : A' \rightarrow M \). We only treat the third case as the previous two may be handled in the same way. In this case \((a,b) \in A^*\) if and only if \((a,b') \in (A')^*\) and \(f^*(a,b') < b_k\). Let \( D \) be the set of \( a \in M^I \) such that \((a,b') \in (A')^*\). An application of the inductive hypothesis shows \( D \) is definable. Let \( \sim \) be the equivalence relation on \( D \) given by \( e \sim d \) if and only if \( f^*(e,b') = f^*(d,b') \). The inductive hypothesis shows \( \sim \) is definable. Applying the elimination of imaginaries for o-minimal expansions of ordered groups we suppose \( D/\sim \) is a definable set \( P \) and let \( \rho : D \rightarrow P \) be the quotient map. We put a relation \( \preceq \) on \( D \) by declaring \( e \preceq d \) if and only if \( f^*(e,b') \leq f^*(d,b') \). The inductive hypothesis shows \( \preceq \) is definable. It is easy to see that \( \preceq \) is a quasi-order on \( D \) which pushes forward to a definable linear order on \( P \) under \( \rho \). Abusing notation we let \( \preceq \) the push-forward of \( \preceq \) to \( P \). Let \( W \) be the set of \( d \in P \) for which there is an \( e \in D \) such that \( \rho(e) = d \) and \( f^*(e,b') < b_k \). Then
\[ \{a \in M^I : (a,b) \in A^*\} = \{a \in M^I : [a \in D] \wedge [\rho(a) \in W]\}. \]
It is easy to see that \( W \) is a cut in \((P,\preceq)\), it follows by Proposition 3.1 that \( W \) is definable. 

\[ \square \]

References