INTERPOLATIVE FUSIONS

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Abstract. We define the interpolative fusion of multiple theories over a common reduct, a notion that aims to provide a general framework to study model-theoretic properties of structures with randomness. In the special case where the theories involved are model complete, their interpolative fusion is precisely the model companion of their union. Several theories of model-theoretic interest are shown to be canonically bi-interpretable with interpolative fusions of simpler theories. We initiate a systematic study of interpolative fusions by also giving general conditions for their existence and relating their properties to those of the individual theories from which they are built.

Contents

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1. Introduction

Throughout the introduction, L_1 and L_2 are first-order languages with the same sorts, $L_\cap = L_1 \cap L_2$, and $L_\cup = L_1 \cup L_2$. We let T_1 and T_2 be L_1 and L_2-theories, respectively, with a common set $T_\cap$ of $L_\cap$-consequences, and $T_\cup = T_1 \cup T_2$. Finally, $\mathcal{M}_\cup$ is an $L_\cup$-structure, $\mathcal{M}_\square$ is the $L_\square$-reduct of $\mathcal{M}_\cup$, and $X_\square$ ranges over $\mathcal{M}_\square$-definable sets for $\square \in \{1, 2, \cap\}$. This is a special case of the setting introduced in Section 3.

We say that $\mathcal{M}_\cup$ is **interpolative** if for all $X_1 \subseteq X_2$, there is an $X_\cap$ such that

$$X_1 \subseteq X_\cap \text{ and } X_\cap \subseteq X_2$$

(more symmetrically: for all X_1 and X_2, we have $X_1 \cap X_2 \neq \varnothing$ unless there are $\mathcal{M}_\cap$-definable sets $X^1_\cap$ and $X^2_\cap$ such that $X_1 \subseteq X^1_\cap$, $X_2 \subseteq X^2_\cap$, and $X^1_\cap \cap X^2_\cap = \varnothing$). This notion is an attempt to capture the idea that $\mathcal{M}_1$ and $\mathcal{M}_2$ interact (with respect to definability) in a generic, independent, or random fashion over the reduct $\mathcal{M}_\cap$. Indeed, the above definition says that the only "information" $\mathcal{M}_1$ has about $\mathcal{M}_2$ comes from $\mathcal{M}_\cap$. If the class of interpolative models of $T_\cup$ is elementary with theory $T^*_\cup$, then we say that $T^*_\cup$ is the **interpolative fusion** (of T_1 and T_2 over $T_\cap$). We also say that "$T^*_\cup$ exists" if the class of interpolative $T_\cup$-models is elementary.

The reader may notice similarities with the Craig-Lyndon-Robinson interpolation theorem: for every L_1-formula φ_1 and L_2-formula φ_2 for which $\models \varphi_1 \to \varphi_2$, there is an $L_\cap$-formula $\varphi_\cap$ such that $\models \varphi_1 \to \varphi_\cap$ and $\models \varphi_\cap \to \varphi_2$. The resemblance is consequential. It allows us to prove the following theorem in Section 3:

Theorem 1.1. *Suppose T_1 and T_2 are model complete. Then $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if it is existentially closed in the class of $T_\cup$-models. Hence, $T^*_\cup$ is precisely the model companion of $T_\cup$, if either of these exists.*

This paper begins a systematic study of interpolative fusions with three aspects: examples, existence conditions, and preservation results. A number of theories of interest may be realized as interpolative fusions of simpler theories. We describe several examples in this paper, and we will describe several more in the sequel [KTW]. We also give general "pseudo-topological" conditions on $T_\cap$ and the T_i which ensure that $T^*_\cup$ exists. When these conditions are satisfied we obtain a pseudo-topological axiomization of $T^*_\cup$. Finally, we seek to understand model-theoretic properties of the interpolative fusion $T^*_\cup$ in terms of properties of its reducts $T_1, T_2, T_\cap$, by proving results of the form "if T_1 and T_2 satisfy P (and possibly some extra additional conditions), then $T^*_\cup$ also satisfies P". Various results about concrete properties of specific theories can then be understood as special cases of results on preservation (or lack of preservation) of model-theoretic properties under interpolative fusion.

1.1. **Examples.** In Section 6, we show that many first-order theories of model-theoretic interest can be put into the common framework of interpolative fusions.

We show in Section 6.1 that if P is an infinite and co-infinite unary predicate on a one-sorted structure $\mathcal{M}$ with underlying set M, then P is a generic predicate as defined in [CP98] if and only if $(\mathcal{M}; P)$ is a model of the interpolative fusion of the theories of $\mathcal{M}$ and $(M; P)$ over the theory of the pure set M.

Let σ be an automorphism of a model complete L-structure $\mathcal{M}$, $\mathcal{N}$ another L-structure, and τ an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. Let T be the theory of $\mathcal{M}$ and T_{Aut} be the theory of a T-model expanded by an L-automorphism. We show in Section 6.2 that $(\mathcal{M}, \mathcal{N}; \tau)$ and $(\mathcal{M}, \mathcal{N}; \tau \circ \sigma)$ are both canonically bi-interpretable with $\mathcal{M}$ and $(\mathcal{M}, \mathcal{N}; \tau, \tau \circ \sigma)$ is canonically bi-interpretable with $(\mathcal{M}; \sigma)$. Further, $(\mathcal{M}; \sigma)$ is existentially closed in the collection of T_{Aut}-models if and only if $(\mathcal{M}, \mathcal{N}; \tau, \tau \circ \sigma)$ is an interpolative structure. It follows that if T_{Aut} has a model companion T^*_{Aut}, then T^*_{Aut} is bi-interpretable with the interpolative fusion of two theories, each of which is bi-interpretable with T.

As a special case of the remarks in the preceding paragraph, we see that the model companion ACFA of the theory of an algebraically closed field equipped with an automorphism is bi-interpretable with an interpolative fusion of two theories, each of which is bi-interpretable with the theory of algebraically closed fields. We show that the same statement holds for the theory DCF of differentially closed fields. The general algebraic framework of D-rings, developed by Moosa and Scanlon [MS14], gives a way of uniformly handling both ACFA and DCF. We show in Section 6.4 that the model companion of the theory of D-rings (for a fixed base ring D) is always bi-interpretable with an interpolative fusion of two theories, each of which is bi-interpretable with the theory of algebraically closed fields.

Let K be an algebraically closed field and v_1, v_2 be non-trivial valuations which induce distinct topologies on K. It follows from [vdD] or [Joh16] that $(K; v_1, v_2)$ is a model of the interpolative fusion of the theories of $(K; v_1)$ and $(K; v_2)$ over the theory of K (see Section 6.5). Now let $\mathbb{F}$ be an algebraic closure of a finite field, $\mathbb{F}^\times$ the multiplicative group of $\mathbb{F}$ (considered as as pure abelian group), and Σ the restriction of the graph of addition on $\mathbb{F}^\times$. Let $\lhd$ be any circular ordering on $\mathbb{F}^\times$ that respects multiplication. Then $(\mathbb{F}^\times; \Sigma, \lhd)$ is a model of the interpolative fusion of the theories of $(\mathbb{F}^\times; \Sigma)$ and $(\mathbb{F}^\times; \lhd)$ over the theory of $\mathbb{F}^\times$. Note that $(\mathbb{F}^\times; \lhd)$ is a cyclically ordered abelian group and $(\mathbb{F}^\times; \Sigma)$ is bi-interpretable with $\mathbb{F}$ (see Section 6.7). The initial motivation of this paper was to find a common generalization of these two examples.

Another interesting source of examples is the expansion of a structure by a generic substructures of a reduct, recently described by d'Elbée [d'E18]. We will discuss this example and others in [KTW].

1.2. **Existence results.** In general $T^*_\cup$ need not exist, and the existence of $T^*_\cup$ may even involve classification-theoretic issues. For example, it is conjectured that if T is unstable then T_{Aut} does not have a model companion. In Section 5 we give general "pseudo-topological" conditions on $T_1, T_2, T_\cap$ which ensure the existence of $T^*_\cup$. The pseudo-topological conditions are highly non-trivial, but they are satisfied in many interesting examples. We also give a natural set of pseudo-topological axioms for $T^*_\cup$ when the pseudo-topological conditions are satisfied. In several concrete examples, the pseudo-topological axioms are essentially identical with known axiomatizations. We now briefly describe these conditions and axioms.

Suppose that dim is an ordinal-valued notion of dimension on definable subsets of $T_\cap$-models satisfying some minimal conditions given in Section 5.1. Most tame theories come with a canonical dimension. We say that an arbitrary set A is pseudo-dense in $X_\cap$ if A intersects every $\mathcal{M}_\cap$-definable $Y_\cap \subseteq X_\cap$ such that $\dim Y_\cap = \dim X_\cap$. We say that $X_\cap$ is a pseudo-closure of A if A is pseudo-dense in $X_\cap$ and $A \subseteq X_\cap$. We say that $\mathcal{M}_i$ is **approximable over** $\mathcal{M}_\cap$ if every $\mathcal{M}_i$-definable set has a pseudo-closure, and we say T_i is **approximable over** $T_\cap$ if the same situation holds for every T_i-model. Then T_i satisfies the pseudo-topological conditions if T_i is approximable over $T_\cap$ and T_i defines pseudo-denseness (see Section 5.1 for a precise definition of the latter). If T_1 and T_2 satisfy the pseudo-topological conditions, then $\mathcal{M}_\cup$ is interpolative if and only if $X_1 \cap X_2 \neq \varnothing$ whenever X_1 and X_2 are both pseudo-dense in some $X_\cap$. The definability of pseudo-denseness ensures this property is axiomatizable.

The use of the term "pseudo-topological" is motivated by consideration of the case, treated in Section 5.3, when $T_\cap$ is o-minimal and dim is the canonical o-minimal dimension. In this case, any theory extending $T_\cap$ defines pseudo-denseness. Furthermore T_i is approximable over $T_\cap$ if and only if $T_\cap$ is an *open core* of T_i, i.e. the closure of any $\mathcal{M}_i$-definable set in any T_i-model $\mathcal{M}_i$ is already $\mathcal{M}_\cap$-definable. This leads to the following:

Theorem 1.2. *Suppose $T_\cap$ is o-minimal. If $T_\cap$ is an open core of both T_1 and T_2 then $T^*_\cup$ exists.*

In the case when $L_\cap$ is $\{=\}$ and $T_\cap$ is the theory of an infinite set with equality, the notion of interpolative fusion is essentially known. It was studied by Winkler in his thesis under Robinson and Macintyre [Win75]. Winkler shows that $T^*_\cup$ exists if only if T_1 and T_2 both eliminate $\exists^\infty$. In Section 5.4, we show that if $T_\cap$ is $\aleph_0$-stable, and dim is Morley rank, then any theory extending $T_\cap$ is approximable over $T_\cap$ (e.g. if $T_\cap$ is the theory of algebraically closed fields, then this follows from the fact that every Zariski closed set is definable). In Section 5.5, we show that if $T_\cap$ is $\aleph_0$-stable, $\aleph_0$-categorical, and weakly eliminates imaginaries, then T_i defines pseudo-denseness if and only if T_i eliminates $\exists^\infty$. This yields a generalization of Winkler's theorem:

Theorem 1.3. *Suppose that $T_\cap$ is $\aleph_0$-stable, $\aleph_0$-categorical, and weakly eliminates imaginaries. If T_1 and T_2 both eliminate $\exists^\infty$, then $T^*_\cup$ exists.*

Amusingly, Theorem 1.3 may also be used prove the other principal result of Winkler's thesis: the existence of generic Skolemizations of model complete theories eliminating $\exists^\infty$. We explain in Section 6.1.2.

In [vdD, 3.1.20] van den Dries notes a similarity between his main result and Winkler's theorem and claims that this similarity *"... suggests a common generalization of Winkler's and my results".* The results of the present paper do not provide such a generalization, as our results do not in fact generalize the main result of [vdD]. However, we believe the present paper provides a moral answer to this suggestion. The fact that we have not obtained the common generalization suggested by van den Dries suggests that our theory is not yet in its final form.

1.3. **Preservation results.** Suppose that $T_\cup$ has a model companion $T^*_\cup$. The examples described above motivate the following question:

How are the model-theoretic properties of $T^*_\cup$ determined by T_1, T_2, and $T_\cap$?

The philosophy is that model-theoretic properties of $T^*_\cup$ should be largely determined by how T_i relates to $T_\cap$ for $i \in \{1,2\}$, and not by any relationship between T_1 and T_2. In the current paper we focus on syntactic tameness properties of $T^*_\cup$ (strengthenings of model-completeness). We describe a general framework for handling strengthenings of model completeness in Section 2.2 and prove syntactic preservation results in Section 4. One of the most important is the following, see Proposition 4.11 below.

Theorem 1.4. *Suppose $T_\cap$ is stable with weak elimination of imaginaries. Suppose $T^*_\cup$ exists. Then every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*

$$\exists y\, (\varphi_1(x,y) \wedge \varphi_2(x,y))$$

where $\varphi_i(x,y)$ is an L_i-formula for $i \in \{1,2\}$ and $(\varphi_1(x,y) \wedge \varphi_2(x,y))$ is bounded in y, i.e. there exists k such that $T^*_\cup \models \forall x\, \exists^{\leq k} y\, (\varphi_1(x,y) \wedge \varphi_2(x,y))$.

This result is close to optimal, as $L_\cup$-formulas are in general not $T^*_\cup$-equivalent to Boolean combinations of L_1 and L_2-formulas. However, in Proposition 4.13, we show that certain restrictive conditions on algebraic closure in T_1 and T_2 do imply that every $L_\cup$-formula is $T^*_\cup$-equivalent to a Boolean combination of L_1 and L_2-formulas. If this special situation holds, and if T_1 and T_2 are both stable (NIP), then $T^*_\cup$ must also be stable (NIP), see Section 4.5. Proposition 4.13 is motivated by some comments in the introduction of [MS14] on the failure of quantifier elimination in ACFA (and in other D-rings).

These syntactic preservation results can be applied to obtain classification-theoretic preservation results which relate the (neo)stability theoretic properties of $T^*_\cup$ to those of $T_1, T_2, T_\cap$. In the present paper we only address preservation of stability and NIP (Section 4.5). We will address preservation of simplicity, NSOP_1, and NTP_2 in the next paper [KTW].

1.4. **Conventions and notation.** Throughout m, and n range over the set of natural numbers (which contains 0) and k and l range over the set of integers. We work in multi-sorted first-order logic. Our semantics allows empty sorts and empty structures. Our syntax includes logical constants $\top$ and $\bot$ interpreted as true and false, respectively. We view constant symbols as 0-ary function symbols.

Throughout, L is a language. Suppose $\mathcal{M}$ is an L-structure. If S is the set of sorts in L, we write $M = (M_s)_{s\in S}$ for the S-indexed family of underlying sets of the sorts of $\mathcal{M}$. If $x = (x_j)_{j\in J}$ is a tuple of variables (possibly infinite), we let $M^x = \prod_{j\in J} M_{s(x_j)}$ where $s(x_j)$ is the sort of the variable x_j. If $\phi(x,y)$ is an L-formula and $b \in M^y$, we let $\phi(M^x, b)$ be the set defined in $\mathcal{M}$ by the $L(b)$-formula $\phi(x,b)$.

When the structure in question is the monster model for a complete theory, we use the same conventions, but write $\boldsymbol{\mathcal{M}}$ instead of $\mathcal{M}$ and $\boldsymbol{M}$ instead of M. When discussing a monster model, we adopt the usual convention that all models of $\mathrm{Th}(\boldsymbol{\mathcal{M}})$ are small elementary substructures of $\boldsymbol{\mathcal{M}}$, and all sets of parameters are small subsets of $\boldsymbol{M}$.

We often work with multiple languages with the same set of sorts. In these cases, we use tuples of variables without specifying the language and define the union and intersection of the languages in the obvious manner. Whenever we consider multiple reducts of a structure, we decorate these reducts with the same decorations as their languages. For example, if $L_0 \subseteq L_1$ are languages, we denote an L_1-structure by $\mathcal{M}_1$, and we denote its reduct $\mathcal{M}_1|_{L_0}$ to L_0 by $\mathcal{M}_0$. In this situation, we write "in $\mathcal{M}_0$" to denote that we are evaluating some concept in the reduct.

2. Preliminaries

In this section we review background material and establish general results for later use which are not specific to the context of interpolative fusions. The reader may skip to Section 3 and refer back to this section as needed.

2.1. Flat formulas.

All concepts in this subsection without further notice are with respect to the language L. A formula is **atomic flat** if it is of the form $x = y$, $R(x_1, \dots, x_n)$, or $f(x_1, \dots, x_n) = y$, where R is an n-ary relation symbol and f is an n-ary function symbol. Here the variables $x, y, x_1, \dots, x_n$ need not be distinct.

A **flat literal** is an atomic flat formula or the negation of an atomic flat formula. The **flat diagram** $\mathrm{fdiag}(\mathcal{A})$ of a structure $\mathcal{A}$ is the set of all flat literal $L(A)$-sentences true in $\mathcal{A}$.

A **flat formula** is a conjunction of finitely many flat literals. An **E♭-formula** is a formula of the form $\exists y\, \varphi(x, y)$, where $\varphi(x, y)$ is flat and $\models \forall x\, \exists^{\leq 1} y\, \varphi(x, y)$.

Remark 2.1. The class of E♭-formulas is closed (up to equivalence) under finite conjunction: the conjunction of the E♭-formulas $\exists y_1\, \varphi_1(x, y_2)$ and $\exists y_2\, \varphi_2(x, y_2)$ is equivalent to the E♭-formula

$$\exists y_1 y_2\, (\varphi_1(x, y_1) \wedge \varphi_2(x, y_2)).$$

The following lemma essentially appears as Theorem 2.6.1 in [Hod93]. Note Hodges uses the term "unnested" instead of "flat".

Lemma 2.2. *Every literal (atomic or negated atomic formula) is logically equivalent to an E♭-formula.*

Proof. We first show that for any term $t(x)$, with variables $x = (x_1, \dots, x_n)$, there is an associated E♭-formula $\varphi_t(x, y)$ such that $\varphi_t(x, y)$ is logically equivalent to $t(x) = y$. We apply induction on terms. For the base case where $t(x)$ is the variable x_k, we let $\varphi_t(x, y)$ be $x_k = y$. Now suppose $t_1(x), \dots, t_m(x)$ are terms and f is an m-ary function symbol. The E♭-formula associated to $f(t_1(x), \dots, t_m(x))$ is

$$\exists z_1 \dots z_m \left[\bigwedge_{i=1}^{m} \varphi_{t_i}(x, z_i) \wedge (f(z_1, \dots, z_m) = y) \right].$$

We now show that every atomic or negated atomic formula is equivalent to an E♭-formula. Suppose $t_1(x), \dots, t_m(x)$ are terms and R is either an m-ary relation symbol or $=$ (in the latter case, we have $m = 2$). Then the atomic formula $R(t_1(x), \dots, t_m(x))$ is equivalent to

$$\exists y_1 \dots \exists y_m \left[\bigwedge_{i=1}^{m} \varphi_{t_i}(x, y_i) \wedge R(y_1, \dots, y_m) \right].$$

Negated atomic formulas can be treated similarly. □

Corollary 2.3. *Every quantifier-free formula is logically equivalent to a finite disjunction of E*$\flat$*-formulas.*

Proof. Suppose $\varphi(x)$ is quantifier-free. Then $\varphi(x)$ is equivalent to a formula in disjunctive normal form, i.e., a finite disjunction of finite conjunctions of literals. Applying Lemma 2.2 to each literal and using Remark 2.1, we find that $\varphi(x)$ is equivalent to a finite disjunction of E$\flat$-formulas. □

2.2. $\mathcal{K}$**-completeness.** Througout this subsection, we fix an L-theory T and let $\mathcal{K}$ be a class of substructures of T-models containing the class of T-models. Then T is $\mathcal{K}$**-complete** if any embedding from a structure in $\mathcal{K}$ to a T-model is partial elementary: if $\mathcal{A} \in \mathcal{K}$ is a substructure of $\mathcal{M} \models T$, and $f\colon \mathcal{A} \to \mathcal{N} \models T$ is an embedding, then $\mathcal{M} \models \varphi(a)$ if and only if $\mathcal{N} \models \varphi(f(a))$ for any formula $\varphi(x)$ and any $a \in A^x$.

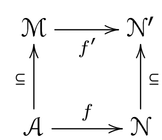
Remark 2.4. The terminology $\mathcal{K}$-complete comes from the following equivalent definition: T is $\mathcal{K}$-complete if and only if for all $\mathcal{A} \in \mathcal{K}$ with $\mathcal{A} \subseteq \mathcal{M} \models T$,

$$T \cup \mathrm{fdiag}(A) \models \mathrm{Th}_{L(A)}(\mathcal{M}),$$

i.e., $T \cup \mathrm{fdiag}(A)$ is a complete $L(A)$-theory. Indeed, if $\mathcal{M}$ is an $L(A)$-structure, then $\mathcal{M} \models \mathrm{fdiag}(\mathcal{A})$ if and only if the obvious map $\mathcal{A} \to \mathcal{M}$ is an embedding.

Suppose T is $\mathcal{K}$-complete. If $\mathcal{K}$ is the class of T-models, then T is **model-complete**. We say T is **substructure-complete** when $\mathcal{K}$ is the class of substructures of T-models. If cl is a closure operator on T-models and $\mathcal{K}$ is the class of cl-closed substructures of T-models, then we say T is **cl-complete**.

The class of T-models has the **amalgamation property over** $\mathcal{K}$ if whenever $\mathcal{A} \in \mathcal{K}$, $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A}$ is a substructure of $\mathcal{M}$, and $f\colon \mathcal{A} \to \mathcal{N}$ is an embedding, then there is an extension $\mathcal{N}'$ of $\mathcal{N}$ and an embedding $f'\colon \mathcal{M} \to \mathcal{N}'$ such that $f'|_A = f$, i.e., the following diagram commutes:



If, in the situation above, we can always choose $\mathcal{N}'$ and f' such that

$$f'(M) \cap N = f'(A) = f(A),$$

then the class of T-models has the **disjoint amalgamation property over** $\mathcal{K}$.

Theorem 2.5. *The theory T is $\mathcal{K}$-complete if and only if T is model-complete and the class of T-models has the amalgamation property over $\mathcal{K}$. Further, if T is $\mathcal{K}$-complete, then every structure in $\mathcal{K}$ is algebraically closed if and only if the class of T-models has the disjoint amalgamation property over $\mathcal{K}$.*

Proof. We prove the first equivalence. Suppose T is $\mathcal{K}$-complete. Then T is model-complete, as every T-model is in $\mathcal{K}$. The amalgamation property over $\mathcal{K}$ follows from [Hod93, Theorem 6.4.1].

Conversely, suppose T is model-complete and the class of T-models has the amalgamation property over $\mathcal{K}$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are T-models, $\mathcal{A} \subseteq \mathcal{M}$ is in $\mathcal{K}$, and $f\colon \mathcal{A} \to \mathcal{N}$ is an embedding. Then there is an extension N' of N and an embedding $f'\colon \mathcal{M} \to \mathcal{N}'$ such that $f'|_A = f$. Since T is model-complete, $\mathcal{N} \preceq \mathcal{N}'$ and f' is an elementary embedding. For any L-formula $\varphi(x)$ and $a \in A^x$, $\mathcal{M} \models \varphi(a)$ if and only if $\mathcal{N}' \models \varphi(f'(a))$ if and only if $\mathcal{N} \models \varphi(f(a))$. So f is partial elementary. Thus T is $\mathcal{K}$-complete.

Now, assuming T is $\mathcal{K}$-complete, we prove the second equivalence. If every structure in $\mathcal{K}$ is algebraically closed, then the class of T-models has the disjoint amalgamation property over $\mathcal{K}$, by [Hod93, Theorem 6.4.5].

Conversely, suppose the class of T-models has the disjoint amalgamation property over $\mathcal{K}$. Assume towards a contradiction that $\mathcal{A} \subseteq \mathcal{M}$ is in $\mathcal{K}$ and A is not algebraically closed in $\mathcal{M}$. Then there is some $c \in M \setminus A$ such that $\text{tp}(c/A)$ has exactly k realizations $c_1, \ldots, c_k$ in $M \setminus A$. Taking $\mathcal{N} = \mathcal{M}$ and $f = \text{id}_A$ in the disjoint amalgamation property, by model-completeness there is an elementary extension $\mathcal{M} \preceq \mathcal{M}'$ and an elementary embedding $f' \colon \mathcal{M} \to \mathcal{M}'$ which is the identity on A and satisfies $f'(M) \cap M = A$. Then $\text{tp}(c/A)$ has $2k$ distinct realizations $c_1, \ldots, c_k, f'(c_1), \ldots, f'(c_k)$ in $\mathcal{M}'$, contradiction. □

We recall some classical facts about model-completeness and model companions.

Fact 2.6 ([Hod93], Theorem 6.5.9, Exercise 6.5.5)**.** The following are equivalent:

(1) T admits an $\forall\exists$-axiomatization.
(2) The class of T-models is closed under unions of chains.
(3) The class of T-models is closed under directed colimits (in the category of L-structures and embeddings).

If one of the above equivalent conditions are satisfied, we say that T is **inductive**.

Fact 2.7 ([Hod93], Theorem 8.3.3)**.** Every model-complete theory is inductive.

An L-theory T^* is a **model companion** of T if T^* is model-complete, every T-model embeds into a T^*-model, and every T^*-model embeds into a T-model.

Fact 2.8 ([Hod93], Theorem 8.2.1, Theorem 8.3.6)**.** Suppose T is inductive. Then:

(1) Every T-model embeds into an existentially closed T-model.
(2) T has a model companion if and only if the class of existentially closed T-models is elementary.
(3) If T has a model companion T^*, then T^* is the theory of existentially closed T-models.

Model-completeness has a syntactic equivalent: every L-formula is T-equivalent to an existential (hence also universal) formula [Hod93, Theorem 8.3.1(e)].

Substructure-completeness also has a syntactic equivalent: quantifier elimination. This follows from [Hod93, Theorem 8.4.1] and Theorem 2.5 above.

Many of the theories we consider are acl-complete. Unfortunately, there does not seem to be a natural syntactic equivalent to acl-completeness. We introduce a slightly stronger notion, bcl-completeness, which does have a syntactic equivalent.

An L-formula $\varphi(x, y)$ is **bounded in** y with bound k (with respect to T) if

$$T \models \forall x\, \exists^{\leq k} y\, \varphi(x, y).$$

A formula $\exists y\, \psi(x, y)$ is **boundedly existential (b.e.)** (with respect to T) if $\psi(x, y)$ is quantifier-free and bounded in y. We allow y to be the empty tuple of variables, so every quantifier-free formula is b.e. (with bound $k = 1$, by convention). The E$\flat$-formulas from Section 2.1 are also b.e. with bound $k = 1$ with respect to the empty theory.

Remark 2.9. The class of b.e. formulas is closed (up to T-equivalence) under conjunction: if $\exists y\, \psi_1(x, y_1)$ and $\exists y_2\, \psi_2(x, y_2)$ are b.e. with bounds k_1 and k_2 on y_1 and y_2 respectively, then

$$(\exists y_1\, \psi_1(x, y_1)) \wedge (\exists y_2\, \psi_2(x, y_2))$$

is T-equivalent to

$$\exists y_1 y_2\, (\psi_1(x, y_1) \wedge \psi_2(x, y_2)),$$

which is b.e. with bound $k_1 \cdot k_2$ on $y_1 y_2$.

Suppose $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. The **boundedly existential algebraic closure** of A in $\mathcal{M}$, denoted $\operatorname{bcl}(A)$, is the set of all b in M such that $\mathcal{M} \models \exists z\, \varphi(a, b, z)$ for some quantifier-free L-formula $\varphi(x, y, z)$ bounded in yz and some $a \in A^x$.

Remark 2.10. Note $\varphi(x, y, z)$ is bounded in yz if and only if it is bounded in z and $\exists z\, \varphi(x, y, z)$ is bounded in y. As a consequence, $b \in \operatorname{bcl}(A)$ if and only if b satisfies a b.e. formula $\exists z\, \varphi(y, z)$ with parameters from A, which is bounded in y. Such a formula is algebraic, so $\operatorname{bcl}(A) \subseteq \operatorname{acl}(A)$.

Lemma 2.11. *If $A \subseteq \mathcal{M}$ then $\langle A \rangle \subseteq \operatorname{bcl}(A)$. Furthermore,* bcl *is a closure operator.*

Proof. Fix $A \subseteq \mathcal{M}$. Suppose $b \in \langle A \rangle$. Then $t(a) = b$ for a term $t(x)$ and a tuple a from A. Then the formula $t(x) = y$ is b.e. (taking z to be the empty tuple of variables) and bounded in y (with bound 1), so it witnesses $b \in \operatorname{bcl}(A)$ by Remark 2.10.

It follows that $A \subseteq \operatorname{bcl}(A)$, and it is clear that $A \subseteq B$ implies $\operatorname{bcl}(A) \subseteq \operatorname{bcl}(B)$. It remains to show bcl is idempotent.

Suppose $b \in \operatorname{bcl}(\operatorname{bcl}(A))$. Then $\mathcal{M} \models \exists z\, \varphi(a, b, z)$ for some quantifier-free formula $\varphi(x, y, z)$ which is bounded in yz and some tuple $a = (a_1, \ldots, a_n)$ from $\operatorname{bcl}(A)$. For each $1 \leq j \leq n$, since a_j is in $\operatorname{bcl}(A)$, $\mathcal{M} \models \exists z_j\, \psi_j(d_j, a_j, z_j)$ for some quantifier-free formula $\psi_j(w_j, x_j, z_j)$ which is bounded in $x_j z_j$, and some tuple d_j from A.

Then the quantifier-free formula

$$\left(\bigwedge_{j=1}^{n} \psi_j(w_j, x_j, z_j) \right) \wedge \varphi(x_1, \ldots, x_n, y, z)$$

is bounded in $x_1 \ldots x_n y z_1 \ldots z_n z$ (by the product of the bounds for φ and the ψ_j), and

$$\mathcal{M} \models \exists x_1 \ldots x_n z_1 \ldots z_n z \left(\bigwedge_{j=1}^{n} \psi_j(d_j, x_j, z_j) \right) \wedge \varphi(x_1, \ldots, x_n, b, z),$$

so $b \in \operatorname{bcl}(A)$. □

Remark 2.12. Every model is acl-closed, every acl-closed set is bcl-closed, and every bcl-closed set is a substructure, therefore:

QE $\Leftrightarrow$ substructure-complete $\Rightarrow$ bcl-complete $\Rightarrow$ acl-complete $\Rightarrow$ model-complete.

Theorem 2.13 clarifies the relationship between acl- and bcl-completeness and provides the promised syntactic equivalent to bcl-completeness.

Theorem 2.13. *The following are equivalent:*

(1) *Every L-formula is T-equivalent to a finite disjunction of b.e. formulas.*
(2) *T is* acl-*complete and* acl = bcl *in T-models.*
(3) *T is* bcl-*complete.*

Proof. We assume (1) and prove (2). We first show acl and bcl agree. Suppose $A \subseteq \mathcal{M} \models T$ and $b \in \operatorname{acl}(A)$, witnessed by an algebraic formula $\varphi(a, y)$ with parameters a from A. Suppose there are exactly k tuples in M^y satisfying $\varphi(a, y)$. Let $\varphi'(x, y)$ be the formula

$$\varphi(x, y) \wedge \exists^{\leq k} y' \, \varphi(x, y'),$$

and note $\varphi'(x, y)$ is bounded in y. By assumption, $\varphi'(x, y)$ is equivalent to a finite disjunction of boundedly existential formulas, so there is some boundedly existential formula $\psi(x, y)$ such that $T \models \psi(x, y) \to \varphi'(x, y)$ and $\mathcal{M} \models \psi(a, b)$. Since $\varphi'(x, y)$ is bounded in y, so is $\psi(x, y)$, and hence $b \in \operatorname{bcl}(A)$ by Remark 2.10.

We continue to assume (1) and show T is acl-complete. Suppose $\mathcal{A}$ is an algebraically closed substructure of $\mathcal{M} \models T$ and $f : \mathcal{A} \to \mathcal{N} \models T$ is an embedding. We show that for any formula $\varphi(x)$, if $\mathcal{M} \models \varphi(a)$, where $a \in A^x$, then $\mathcal{N} \models \varphi(f(a))$. By our assumption, $\varphi(x)$ is equivalent to a finite disjunction of boundedly existential formulas, so there is some boundedly existential formula $\exists y \, \psi(x, y)$ such that

$$T \models (\exists y \, \psi(x, y)) \to \varphi(x) \quad \text{and} \quad \mathcal{M} \models \exists y \, \psi(a, y).$$

Let $b \in M^y$ be a witness for the existential quantifier. Then each component of the tuple b is in $\operatorname{acl}(a) \subseteq A$, since A is algebraically closed. And ψ is quantifier-free, so $\mathcal{N} \models \psi(f(a), f(b))$, and hence $\mathcal{N} \models \varphi(f(a))$.

It is clear that (2) implies (3).

We now assume (3) and prove (1). For any finite tuple of variables x, let Δ_x be the set of boundedly existential formulas with free variables from x.

Claim: For all models $\mathcal{M}$ and $\mathcal{N}$ of T and all tuples $a \in M^x$ and $a' \in N^x$, if $\operatorname{tp}_{\Delta_x}(a) \subseteq \operatorname{tp}_{\Delta_x}(a')$, then $\operatorname{tp}(a) = \operatorname{tp}(a')$.

Proof of claim: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are models of T, $a \in M^x$, $a' \in N^x$, and $\operatorname{tp}_{\Delta_x}(a) \subseteq \operatorname{tp}_{\Delta_x}(a')$. Let y be a tuple of variables enumerating the elements of $\operatorname{bcl}(a)$ which are not in a. Let $p(x, y) = \operatorname{qftp}(\operatorname{bcl}(a))$, and let $q(x) = \operatorname{tp}(a')$. We claim that $T \cup p(x, y) \cup q(x)$ is consistent.

Let $b = (b_1, \ldots, b_n)$ be a finite tuple from $\operatorname{bcl}(a)$ which is disjoint from a, and let $\psi(x, y')$ be a quantifier-free formula such that $\mathcal{M} \models \psi(a, b)$ (where $y' = (y_1, \ldots, y_n)$ is the finite subtuple of y enumerating b).

For each $1 \leq j \leq n$, the fact that $b_j \in \operatorname{bcl}(a)$ is witnessed by $\mathcal{M} \models \exists z_j \, \varphi_j(a, b_j, z_j)$, where $\varphi_j(x, y_j, z_j)$ is quantifier-free and bounded in $y_j z_j$. Letting $z = (z_1, \ldots, z_n)$, the conjunction $\bigwedge_{j=1}^{n} \varphi_j(x, y_j, z_j)$ is a quantifier-free formula $\varphi(x, y', z)$ which is bounded in $y'z$. It follows that $\varphi(x, y', z) \wedge \psi(x, y')$ is also bounded in $y'z$, and $\mathcal{M} \models \exists z \, (\varphi(a, b, z) \wedge \psi(a, b))$. Then

$$\exists y' z \, (\varphi(x, y', z) \wedge \psi(x, y')) \in \operatorname{tp}_{\Delta_x}(a) \subseteq \operatorname{tp}_{\Delta_x}(a'),$$

so $\mathcal{N} \models \exists y' z \, (\varphi(a', y', z) \wedge \psi(a', y'))$. Letting $b' \in N_{y'}$ be a witness for the first block of existential quantifiers, $\mathcal{N} \models \psi(a', b')$, so $T \cup \{\psi(x, y')\} \cup q(x)$ is consistent.

By compactness, $T \cup p(x, y) \cup q(x)$ is consistent, so there exists a model $\mathcal{N}' \models T$, a tuple $a'' \in (N')^x$ realizing $q(x)$, and an embedding $f : \operatorname{bcl}(a) \to \mathcal{N}'$ such that $f(a) = a''$. By bcl-completeness, we have $\operatorname{tp}(a) = \operatorname{tp}(a'') = \operatorname{tp}(a')$, as was to be shown.

Having established the claim, we conclude with a standard compactness argument. Let $\varphi(x)$ be an L-formula. Suppose $\mathcal{M} \models T$ and $\mathcal{M} \models \varphi(a)$. Let $p_a(x) = \operatorname{tp}_{\Delta_x}(a)$. By the claim, $T \cup p_a(x) \cup \{\neg\varphi(x)\}$ is inconsistent. Since $p_a(x)$

is closed under finite conjunctions (up to equivalence) by Remark 2.9, there is a formula $\psi_a(x) \in p_a(x)$ such that $T \models \psi_a(x) \to \varphi(x)$.

Now

$$T \cup \{\varphi(x)\} \cup \{\neg\psi_a(x) \mid \mathcal{M} \models T \text{ and } \mathcal{M} \models \varphi(a)\}$$

is inconsistent, so there are finitely many $a_1, \ldots, a_n$ such that $T \models \varphi \to (\bigvee_{i=1}^n \psi_{a_i}(x))$. Since also $T \models (\bigvee_{i=1}^n \psi_{a_i}(x)) \to \varphi(x)$, we have shown that $\varphi(x)$ is T-equivalent to $\bigvee_{i=1}^n \psi_{a_i}(x)$. □

It may be surprising that acl-completeness does not already imply every formula is equivalent to a finite disjunction of b.e. formulas, i.e., acl-completeness is not equivalent to bcl-completeness. We give a counterexample.

Example 2.14. Let L be the language containing a single unary function symbol f. We denote by $E(x, y)$ the equivalence relation defined by $f(x) = f(y)$. We say an element of an L-structure is **special** if it is in the image of f. Let T be the theory asserting the following:

(1) Models of T are non-empty.
(2) There are no cycles, i.e., for all $n \geq 1$, $\forall x\, f^n(x) \neq x$.
(3) Each E-class is infinite and contains exactly one special element.

Every T-model can be decomposed into a disjoint union of **connected components**, each of which is a chain of E-classes, $(C_n)_{n\in\mathbb{Z}}$, such that each class C_n contains a unique special element a_n, and $f(b) = a_n$ for all $b \in C_{n-1}$.

Let A be a subset of a T-model. Then $\mathrm{acl}(A)$ consists of A, together with the $\mathbb{Z}$-indexed chain of special elements in each connected component which meets A. But $\mathrm{bcl}(A)$ is just the substructure generated by A: it only contains the special elements from E-classes further along in the chain than some element of A. Indeed, if a_n is the unique special element in class C_n, $a_n \notin A$, and no element of A is in any class C_m with $m < n$ in the same connected component, then a_n does not satisfy any bounded and b.e. formula with parameters from A.

It is not hard to show that T is acl-complete (and hence complete, since $\mathrm{acl}(\varnothing) = \varnothing$), but not bcl-complete. For an explicit example of a formula which is not equivalent to a disjunction of b.e. formulas, consider the formula

$$\exists y\, f(y) = x$$

defining the special elements.

2.3. **Existential bi-interpretations.** Here we set our notation for interpretations and related notions. We will then show that existential bi-interpretations preserve the property of being existentially closed, and hence restrict to bi-interpretations between model companions, when these exist.

Let T be an L-theory, and let T' be an L'-theory. An **interpretation** of T' in T, $F\colon T \rightsquigarrow T'$, consists of the following data:

(1) For every sort s' in L', an L-formula $\varphi_{s'}(x_{s'})$ and an L-formula $E_{s'}(x_{s'}, x^*_{s'})$.
(2) For every relation symbol R' in L' of type $(s'_1, \ldots, s'_n)$ in L', an L-formula $\varphi_{R'}(x_{s'_1}, \ldots, x_{s'_n})$.
(3) For every function symbol f' in L' of type $(s'_1, \ldots, s'_n) \to s'$ in L', an L-formula $\varphi_{f'}(x_{s'_1}, \ldots, x_{s'_n}, x_{s'})$.

We then require that for every model $\mathcal{M} \vDash T$, the formulas above define an L'-structure $\mathcal{M}'$ in the natural way, such that $\mathcal{M}' \vDash T'$. See [Hod93, Section 5.3] for details. We sometimes denote $\mathcal{M}'$ by $F(\mathcal{M})$. For every sort s' in L', we write $\pi_{s'}$ for the surjective quotient map $\varphi_{s'}(M^{x_{s'}}) \to M'_{s'}$.

An interpretation $F\colon T \rightsquigarrow T'$ is an **existential interpretation** if for each sort s' in L', the L-formula $\varphi_{s'}(x_{s'})$ is T-equivalent to an existential formula, and all other formulas and their negations (i.e., the formulas $E_{s'}$, $\neg E_{s'}$, $\varphi_{R'}$, $\neg\varphi_{R'}$, $\varphi_{f'}$, and $\neg\varphi_{f'}$) are each T-equivalent to an existential formula.

Lemma 2.15. *Suppose $F\colon T \rightsquigarrow T'$ is a existential interpretation. Let $\varphi'(y)$ be a quantifier-free L'-formula, where $y = (y_1, \dots, y_n)$ and y_i is a variable of sort s'_i. Then there is an existential L-formula $\widehat{\varphi}(x_{s'_1}, \dots, x_{s'_n})$ such that for every $\mathcal{M} \vDash T$ and every tuple $a = (a_1, \dots, a_n)$ with $a_i \in \varphi_{s'_i}(M^{x_{s'_i}})$, $\mathcal{M} \vDash \widehat{\varphi}(a)$ if and only if $F(\mathcal{M}) \vDash \varphi'(\pi_{s_1}(a_1), \dots, \pi_{s_n}(a_n))$.*

Proof. By Corollary 2.3, $\varphi'(y)$ is equivalent to a finite disjunction of E$\flat$ formulas. The rest of the proof is as in [Hod93, Theorem 5.3.2]. The fact that the formulas $E_{s'}$, $\neg E_{s'}$, $\varphi_{R'}$, $\neg\varphi_{R'}$, $\varphi_{f'}$, and $\neg\varphi_{f'}$ are existential implies that flat literal L'-formulas can be pulled back to existential L-formulas, and the fact that the formulas φ'_s are existential is used in the inductive step to handle existential quantifiers. □

A **bi-interpretation** (F, G, η, η') between T and T' consists of an interpretation $F\colon T \rightsquigarrow T'$, an interpretation $G\colon T' \rightsquigarrow T$, together with L-formulas and L'-formulas defining for each $\mathcal{M} \vDash T$ and each $\mathcal{N}' \vDash T'$ isomorphisms

$$\eta_{\mathcal{M}} : \mathcal{M} \to G(F(\mathcal{M})) \quad \text{and} \quad \eta'_{\mathcal{N}'} : \mathcal{N}' \to F(G(\mathcal{N}')).$$

See [Hod93, Section 5.4] for the precise definition. Such a bi-interpretation is **existential** if F and G are each existential interpretations, and moreover the aforementioned L-formulas and L'-formulas are existential. If there is an existential bi-interpretation between T and T', we say that T and T' are existentially bi-interpretable. The following is [Hod93, Exercise 5.4.3]:

Lemma 2.16. *Suppose $F\colon T \rightsquigarrow T'$ is existential. Then F induces a functor from the category of models of T and embeddings to the category of models of T' and embeddings. Suppose moreover that (F, G, η, η') is an existential bi-interpretation from T to T'. Then the induced functors form an equivalence of categories; in other words, if $f : \mathcal{M} \to \mathcal{N}$ is an L-embedding, then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\eta_{\mathcal{N}}} & G(F(\mathcal{N})) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle G(F(f))} \\ \mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & G(F(\mathcal{M})) \end{array}$$

We next prove the main result of this subsection:

Proposition 2.17. *Suppose T and T' are existentially bi-interpretable. Then $\mathcal{M}$ is an e.c. model of T if and only if $F(\mathcal{M})$ is an e.c. model of T'.*

Proof. Let (F, G, η, η') be an existential bi-interpretation of between T and T'. It suffices to show that if $F(\mathcal{M})$ is an e.c. model of T', then $\mathcal{M}$ is an e.c. model of T. Indeed, by symmetry it follows that if $G(\mathcal{N}')$ is an e.c. model of T, then $\mathcal{N}'$ is an e.c. model of T'. And then, since $\eta_{\mathcal{M}} : \mathcal{M} \to G(F(\mathcal{M}))$ is an isomorphism, if $\mathcal{M}$ is e.c., then $F(\mathcal{M})$ is e.c.

So assume that $F(\mathcal{M})$ is an e.c. model of T'. Let $f\colon \mathcal{M} \to \mathcal{N}$ be an embedding of T-models, and let $\varphi(y)$ be a quantifier-free formula with parameters from $\mathcal{M}$ which is satisfied in $\mathcal{N}$. Applying Lemma 2.16 and moving the parameters of $\varphi(y)$ into $G(F(\mathcal{M}))$ by the isomorphism $\eta_{\mathcal{M}}$, we find that $\varphi(y)$ is satisfied in $G(F(\mathcal{N}))$, and it suffices to show that it is satisfied in $G(F(\mathcal{M}))$.

By Lemma 2.15, there is an existential L'-formula $\widehat{\varphi}'(x)$ with parameters from $F(\mathcal{M})$ such that $F(\mathcal{N}) \models \widehat{\varphi}'(a)$ if and only if $G(F(\mathcal{N})) \models \varphi(b)$, where b is the image of a under the appropriate π_s quotient maps. Writing $\widehat{\varphi}'(x)$ as $\exists z\, \psi'(x,z)$, we have $F(\mathcal{N}) \models \psi'(a,c)$ for some c, where a is any preimage of the tuple from $G(F(\mathcal{N}))$ satisfying $\varphi(y)$. But since $F(\mathcal{M})$ is e.c., there are some a^* and c^* in $F(\mathcal{M})$ such that $\mathcal{M} \models \psi'(a^*, c^*)$, so $\mathcal{M} \models \widehat{\varphi}'(a^*)$, and it follows that $\varphi(y)$ is satisfied in $G(F(\mathcal{M}))$, as desired. □

Corollary 2.18. *Suppose T and T' are inductive, and T has a model companion T^*. If (F, G, η, η') is an existential bi-interpretation between T and T', then T' has a model companion $(T')^*$, and (F, G, η, η') restricts to an existential bi-interpretation between T^* and $(T')^*$.*

Proof. By [Hod93, Theorem 5.3.2], for every L-sentence $\varphi \in T^*$, there is an L'-sentence φ' such that for all $\mathcal{M}' \models T'$, $\mathcal{M}' \models \varphi'$ if and only if $G(\mathcal{M}) \models \varphi$. Let $(T')^* = T' \cup \{\varphi' \mid \varphi \in T^*\}$. Then $\mathcal{M}' \models (T')^*$ if and only if $\mathcal{M}' \models T'$ and $G(\mathcal{M}') \models T^*$. By Proposition 2.17, $\mathcal{M}' \models (T')^*$ if and only if $\mathcal{M}'$ is an e.c. model of T'. So $(T')^*$ is the model companion of T'. And Proposition 2.17 further implies that $\mathcal{M} \models T^*$ if and only if $F(\mathcal{M}) \models (T')^*$. So (F, G, η, η') restricts to an existential bi-interpretation between the model companions. □

2.4. **Stationary independence relations.** In this subsection, T is a complete L-theory, L' is a first order language extending L, and T' is a complete L'-theory extending T. Let $\boldsymbol{\mathcal{M}}'$ be a monster model of T' and $\boldsymbol{\mathcal{M}}$ be the L-reduct of $\boldsymbol{\mathcal{M}}'$, so $\boldsymbol{\mathcal{M}}$ is a monster model of T.

Let $\mathop{\smile\!\!\!\!\!\mid}$ be a ternary relation on small subsets of $\boldsymbol{\mathcal{M}}$. We consider the following properties that $\mathop{\smile\!\!\!\!\!\mid}$ may satisfy. The first three are specific to T, while the fourth concerns the relationship between T and T'. We let A, B, and C range over arbitrary small subsets of $\boldsymbol{\mathcal{M}}$.

(1) **Invariance:** If σ is an automorphism of $\boldsymbol{\mathcal{M}}$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$ if and only if $\sigma(A) \mathop{\smile\!\!\!\!\!\mid}_{\sigma(C)} \sigma(B)$.
(2) **Algebraic independence:** If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, then
$$\mathrm{acl}_L(AC) \cap \mathrm{acl}_L(BC) = \mathrm{acl}_L(C).$$
(3) **Stationarity (over algebraically closed sets):** If $C = \mathrm{acl}_L(C)$, $\mathrm{tp}_L(A/C) = \mathrm{tp}_L(A^*/C)$, $A \mathop{\smile\!\!\!\!\!\mid}_C B$, and $A^* \mathop{\smile\!\!\!\!\!\mid}_C B$, then $\mathrm{tp}_L(A/BC) = \mathrm{tp}_L(A^*/BC)$.
(4) **Full existence (over algebraically closed sets) in T':** If $C = \mathrm{acl}_{L'}(C)$ then there exists A^* with $\mathrm{tp}_{L'}(A^*/C) = \mathrm{tp}_{L'}(A/C)$ and $A^* \mathop{\smile\!\!\!\!\!\mid}_C B$ in $\boldsymbol{\mathcal{M}}$.

For brevity, we omit the parenthetical "(over algebraically closed sets)" in properties (3) and (4).

We say $\mathop{\smile\!\!\!\!\!\mid}$ is a **stationary independence relation** in T if it satisfies invariance, algebraic independence, and stationarity. In other words, a stationary independence relation $\mathop{\smile\!\!\!\!\!\mid}$ identifies, for every L-type $p(x) \in S_x(C)$ and every set B, a nonempty set of "independent" extensions of $p(x)$ in $S_x(BC)$, so that if A^* realizes an independent extension $p'(x)$ of $p(x)$, then $\mathrm{acl}_L(A^*C) \cap \mathrm{acl}_L(BC) = \mathrm{acl}_L(C)$. Moreover, if $C = \mathrm{acl}_L(C)$, then there is a unique independent extension of $p(x)$ to $S_x(BC)$.

Our definition of a stationary independence relation differs from that introduced in [TZ13]. Most natural stationary independence relations satisfy additional axioms (symmetry, monotonicity, etc.). We only require the axioms listed above.

Forking independence $\mathop{\smile\!\!\!\!\!\mid}^f$ in a stable theory with weak elimination of imaginaries is the most familiar stationary independence relation, and this is the relation we will use in most examples. However, as the next example shows, there are also non-trivial examples in unstable theories.

Example 2.19. Suppose L contains a single binary relation E, and T is the theory of the random graph (the Fraïssé limit of the class of finite graphs). Define:

$$A \mathop{\smile\!\!\!\!\!\mid}^E_C B \iff A \cap B \subseteq C \text{ and } aEb \text{ for all } a \in A \setminus C \text{ and } b \in B \setminus C$$

$$A \mathop{\smile\!\!\!\!\!\mid}^{\not E}_C B \iff A \cap B \subseteq C \text{ and } \neg aEb \text{ for all } a \in A \setminus C \text{ and } b \in B \setminus C.$$

Both $\mathop{\smile\!\!\!\!\!\mid}^E$ and $\mathop{\smile\!\!\!\!\!\mid}^{\not E}$ are stationary independence relations in T.

Now let $L' = \{E, P\}$, where P is a unary predicate, and let T' be the theory of the Fraïssé limit of the class of finite graphs with a predicate P naming a clique. T' extends T and has quantifier elimination, and $\mathrm{acl}_{L'}(A) = A$ for all sets A.

Then $\mathop{\smile\!\!\!\!\!\mid}^E$ has full existence in T'. Indeed, for any A, B, and C, let $p(x) = \mathrm{tp}_{L'}(A/C)$, where $x = (x_a)_{a\in A}$ is a tuple of variables enumerating A. The type $p(x) \cup \{x_a E b \mid a \in A \setminus C \text{ and } b \in B \setminus C\}$ is consistent, and for any realization A^* of this type, we have $A^* \mathop{\smile\!\!\!\!\!\mid}^E_C B$ in $\mathcal{M}$.

On the other hand, let a and b be any two elements of $\mathcal{M}'$ satisfying P. Then for any realization a^* of $\mathrm{tp}_{L'}(a/\varnothing)$, we have $P(a^*)$, so a^*Eb, and $a^* \not\mathop{\smile\!\!\!\!\!\mid}^{\not E}_\varnothing b$ in $\mathcal{M}$. So $\mathop{\smile\!\!\!\!\!\mid}^{\not E}$ does not have full existence in T'.

The remainder of this subsection is devoted to the proof that when T is stable with weak elimination of imaginaries, the stationary independence relation $\mathop{\smile\!\!\!\!\!\mid}^f$ in T always has full existence in T'. We first recall some definitions. T has **stable forking** if whenever a complete type $p(x)$ over B forks over $A \subseteq B$, then there is a stable formula $\delta(x, y)$ such that $\delta(x, b) \in p(x)$ and $\delta(x, b)$ forks over A. Every theory with stable forking is simple; the converse is the Stable Forking Conjecture, which remains open (see [KP00]).

We recall a few variations on the notion of elimination of imaginaries (see [CF04]).

(1) T has **elimination of imaginaries** if every $a \in \mathcal{M}^{\mathrm{eq}}$ is interdefinable with some $b \in \mathcal{M}$, i.e., $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{dcl}^{\mathrm{eq}}(a)$.
(2) T has **weak elimination of imaginaries** if for every $a \in \mathcal{M}^{\mathrm{eq}}$ there is some $b \in \mathcal{M}$ such that $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{acl}^{\mathrm{eq}}(a)$.
(3) T has **geometric elimination of imaginaries** if every $a \in \mathcal{M}^{\mathrm{eq}}$ is interalgebraic with some $b \in \mathcal{M}$, i.e., $a \in \mathrm{acl}^{\mathrm{eq}}(b)$ and $b \in \mathrm{acl}^{\mathrm{eq}}(a)$.

The following lemma is essentially the same idea as [PT97, Lemma 3], which itself makes use of key ideas from [HP94, Lemmas 5.5 and 5.8].

Lemma 2.20. *Suppose T has stable forking and geometric elimination of imaginaries. Then $\mathop{\smile\!\!\!\!\!\!|}^f$ in T has full existence in T'.*

Proof. Suppose towards a contradiction that there exist sets A, B, and C in $\mathcal{M}'$ such that $C = \operatorname{acl}_{L'}(C)$, and for any A^* with $\operatorname{tp}_{L'}(A^*/C) = \operatorname{tp}_{L'}(A/C)$, $A^* \not\mathop{\smile\!\!\!\!\!\!|}^f_C B$ in $\mathcal{M}$. We may assume $C \subseteq B$. Let $p(x) = \operatorname{tp}_{L'}(A/C)$. Since T has stable forking, the fact that $\operatorname{tp}_{L'}(A^*/B)$ forks over C is always witnessed by a stable formula. So the partial type

$$p(x) \cup \{\neg\delta(x,b) \mid \delta(x,y) \in L \text{ is stable, and } \delta(x,b) \text{ forks over } C \text{ in } \mathcal{M}\}$$

is not satisfiable in $\mathcal{M}'$. By saturation and compactness, we may assume that A is finite and x is a finite tuple of variables. And as stable formulas and forking formulas are closed under disjunctions, there is an $L'(C)$-formula $\varphi(x) \in p(x)$ and a stable L-formula $\delta(x,y)$ such that $\delta(x,b)$ forks over C, and

$$\mathcal{M}' \models \forall x\,(\varphi(x) \to \delta(x,b)).$$

Let $S_\delta(\boldsymbol{M})$ be the space of global δ-types, and let $[\varphi]$ be the closure in $S_\delta(\boldsymbol{M})$ of the set of δ-types of tuples a from $\boldsymbol{M}^x$ such that $\mathcal{M}' \models \varphi(a)$. This is the set of all global δ-types $r(x)$ such that $\chi(x) \in r(x)$ whenever $\chi(x)$ is a positive or negative instance of δ and $\varphi(\mathcal{M}') \subseteq \chi(\mathcal{M}')$. Since δ is stable, $[\varphi]$ contains finitely many points of maximal Cantor-Bendixson rank. Let $q(x)$ be such a point. Applying stability of δ again, we see that $q(x)$ is definable by an L-formula $\theta(y,c)$ with parameters c from $\boldsymbol{M}$. That is, $\delta(x,b^*) \in q(x)$ if and only if $\mathcal{M} \models \theta(b^*,c)$. Let $\sim_\theta$ be the equivalence relation given by

$$c_1 \sim_\theta c_2 \quad \text{if and only if} \quad \mathcal{M}' \models \forall y\,[\theta(y,c_1) \leftrightarrow \theta(y,c_2)],$$

and let $[c]_\theta \in \mathcal{M}^{\mathrm{eq}}$ be the $\sim_\theta$-equivalence class of c. Then $[c]_\theta$ is a canonical parameter for q, i.e., an automorphism of $\mathcal{M}$ fixes q if and only if it fixes $\theta(\mathcal{M},c)$, if and only if it fixes $[c]_\theta$.

By geometric elimination of imaginaries, $[c]_\theta$ is interalgebraic with a tuple d from $\mathcal{M}$. In particular, there is an L-formula $\chi(z,w)$ such that for all c^*, $\mathcal{M} \models \chi(c^*,d)$, if and only if $[c^*]_\theta$ is one of the finitely many realizations of $\operatorname{tp}_{L^{eq}}([c]_\theta/d)$.

Note $[\varphi]$ is fixed setwise by any L'-automorphism fixing C, so q has finitely many conjugates under such automorphisms. It follows that $[c]_\theta$ and hence d have finitely many conjugates under such automorphisms. Thus $d \in C$, as C is algebraically closed in $\mathcal{M}'$.

Now $\delta(x,b)$ divides over C in $\mathcal{M}$, since forking and dividing agree in simple theories [Cas11, Prop. 5.17]. This is witnessed by an indiscernible sequence $(b_i)_{i\in\omega}$ in $\operatorname{tp}_L(b/C)$ such that $\{\delta(x,b_i)\}_{i\in\omega}$ is inconsistent. Since $\mathcal{M} \models \delta(a,b)$ for all $a \in \varphi(\mathcal{M}')$, we have $\delta(x,b) \in q(x)$, so $\mathcal{M}' \models \theta(b,c)$, and hence

$$\exists z\,(\chi(z,d) \wedge \theta(y,z)) \in \operatorname{tp}_L(b/C).$$

So we have a sequence $(c_i)_{i\in\omega}$ such that $\chi(c_i,d) \wedge \theta(b_i,c_i)$ for all i. The c_i fall into only finitely many $\sim_\theta$ equivalence classes, so we may refine $(b_i)_{i\in\omega}$ to a subsequence and fix a constant c^* such that $\chi(c^*,d) \wedge \theta(b_i,c^*)$ for all i. But since $[c^*]_\theta$ and $[c]_\theta$ have the same type over d in $\mathcal{M}^{\mathrm{eq}}$, $\theta(y,c^*)$ defines a consistent global δ-type $r(x)$, and $\{\delta(x,b_i)\}_{i\in\omega} \subseteq r(x)$, hence this set is consistent, which is a contradiction. □

Remark 2.21. The following counterexample shows the assumptions of geometric elimination of imaginaries in T and $C = \operatorname{acl}(C)$ in $\mathcal{M}'$ (not just in $\mathcal{M}$) in Lemma 2.20

are necessary. Let T be the theory of an equivalence relation with infinitely many infinite classes. Let T' be the expansion of this theory by a single unary predicate P naming one of the classes. Let a and b be two elements of the class named by P in $\mathfrak{M}'$, and let $C = \varnothing$ (which is algebraically closed in $\mathfrak{M}$ and $\mathfrak{M}'$). For any a^* such that $\mathrm{tp}_{L'}(a^*/\varnothing) = \mathrm{tp}_{L'}(a/\varnothing)$, we have $a^* Eb$, and xEb forks over $\varnothing$ in $\mathfrak{M}$. To fix this, we move to $\mathfrak{M}^{\mathrm{eq}}$, so we have another sort containing names for all the E-classes. Note that $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ in $\mathfrak{M}$ still doesn't contain any of these names. But $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ in $\mathfrak{M}'$ contains the name for the class named by P, since it is fixed by L'-automorphisms. And we recover the lemma, since xEb does not fork over the name for the E-class of b.

Remark 2.22. It is also possible for Lemma 2.20 to fail when there are unstable forking formulas. Let T be be the theory of $(\mathbb{Q}, <)$ and T' be the expansion of T by a unary predicate P defining an open interval (p, p'), where $p < p'$ are irrational reals. Let $b_1 < a < b_2$ be elements of $\mathfrak{M}'$ such that $a \in P$ and $b_1, b_2 \notin P$. Let $C = \varnothing$ (which is algebraically closed in $\mathfrak{M}'$). Then for any realization a^* of $\mathrm{tp}_{L'}(a/\varnothing)$, we have $a^* \not\mathop{\smile\!\!\!\!\!\!|}^f_{\varnothing} b_1 b_2$ in $\mathfrak{M}$, witnessed by the formula $b_1 < x < b_2$.

Remark 2.23. It is not possible to strengthen the conclusion of Lemma 2.20 to the following: For all small sets A, B, and C, such that $C = \mathrm{acl}_{L'}(C)$, and for any A'' such that $\mathrm{tp}_L(A''/C) = \mathrm{tp}_L(A/C)$ and $A'' \mathop{\smile\!\!\!\!\!\!|}^f_C B$ in $\mathfrak{M}$, there exists A' with $\mathrm{tp}_{L'}(A'/C) = \mathrm{tp}_{L'}(A/C)$ and $\mathrm{tp}_L(A'/BC) = \mathrm{tp}_L(A''/BC)$.

That is, while it is possible to find a realization A' of $\mathrm{tp}_{L'}(A/C)$ such that $\mathrm{tp}_L(A'/BC)$ is a nonforking extension of $\mathrm{tp}_L(A/C)$, it is not possible in general to obtain an arbitrary nonforking extension of $\mathrm{tp}_L(A/C)$ in this way.

For a counterexample, consider the theories T and T' from Example 2.19 above. T has stable forking and geometric elimination of imaginaries. Let a and b be elements of the clique defined by P in $\mathfrak{M}'$, and let $C = \varnothing$ (which is algebraically closed in $\mathfrak{M}'$). Let a'' be any element such that $\mathfrak{M}' \models \neg a'' Eb$, and note that $a'' \mathop{\smile\!\!\!\!\!\!|}^f_{\varnothing} b$ and $\mathrm{tp}_L(a''/\varnothing) = \mathrm{tp}_L(a/\varnothing)$ (there is only one 1-type over the empty set with respect to T). But for any a' with $\mathrm{tp}_{L'}(a'/\varnothing) = \mathrm{tp}_{L'}(a/\varnothing)$, $\mathfrak{M}' \models P(a')$, so $a' Eb$, and $\mathrm{tp}_L(a'/b) \neq \mathrm{tp}_L(a''/b)$.

Proposition 2.24. *If T is stable with weak elimination of imaginaries, then $\mathop{\smile\!\!\!\!\!\!|}^f$ is a stationary independence relation in T which has full existence in T'.*

The strengthening of the hypothesis on T from stable forking and geometric elimination of imaginaries in Lemma 2.20 to stability and weak elimination of imaginaries in Proposition 2.24 is due to the requirement that $\mathop{\smile\!\!\!\!\!\!|}^f$ is a stationary independence relation in T.

A simple theory is stable if and only if $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies stationarity over $\mathrm{acl}^{\mathrm{eq}}$-closed sets [Cas11, Ch. 11]. A stable theory has weak elimination of imaginaries if and only if it has geometric elimination of imaginaries and $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies stationarity over acl-closed sets [CF04, Prop. 3.2 and 3.4]. So stability with weak elimination of imaginaries is the natural hypothesis for Proposition 2.24.

3. Interpolative structures and interpolative fusions

Throughout this section, let $L_\cap$ be a language, let $(L_i)_{i\in I}$ be a nonempty family of languages such that L_i has the same set of sorts as $L_\cap$ for $i \in I$, and $L_i \cap L_j = L_\cap$ for all distinct $i, j \in I$, and let $L_\cup = \bigcup_{i\in I} L_i$.

3.1. **Interpolative structures.** Let $\mathcal{M}_\cup$ be an $L_\cup$-structure. Suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$. Then $(X_i)_{i\in J}$ is **separated** if there is a family $(X^i)_{i\in J}$ of $\mathcal{M}_\cap$-definable subsets of M^x such that

$$X_i \subseteq X^i \text{ for all } i \in J, \text{and } \bigcap_{i\in J} X^i = \varnothing.$$

We say $\mathcal{M}_\cup$ is **interpolative** if for all families $(X_i)_{i\in J}$ such that $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$, $(X_i)_{i\in J}$ is separated if and only if $\bigcap_{i\in J} X_i \neq \varnothing$. Note that this generalizes the setting in the introduction.

Remark 3.1. If we change languages in a way that does not change the class of definable sets (with parameters), then the class of interpolative $L_\cup$-structures is not affected. In particular:

(1) An interpolative structure $\mathcal{M}_\cup$ remains so after adding new constant symbols which name any subset of M to all the languages $L_\Box$ for $\Box \in I \cup \{\cup, \cap\}$.
(2) Suppose $L_i^\diamond$ is a definitional expansion of L_i for each $i \in I$ and $L_\cup^\diamond = \bigcup_{i\in I} L_i^\diamond$ is the resulting definitional expansion of $L_\cup$. Then any $L_\cup$-structure $\mathcal{M}_\cup$ has a canonical expansion $\mathcal{M}_\cup^\diamond$ to an $L_\cup^\diamond$-structure. And $\mathcal{M}_\cup$ is an interpolative $L_\cup$-structure if and only if $\mathcal{M}_\cup^\diamond$ is an interpolative $L_\cup^\diamond$-structure.
(3) An interpolative $\mathcal{M}_\cup$-structure remains so after replacing each function symbol f in each of the languages $L_\Box$ for $\Box \in I \cup \{\cup, \cap\}$ by a relation symbol R_f, interpreted as the graph of the interpretation of f in $\mathcal{M}_\cup$.
(4) Suppose $\mathcal{M}_\cup$ is an $L_\cup$-structure. Moving to $\mathcal{M}_\cap^{\mathrm{eq}}$ involves the introduction of new sorts and function symbols for quotients by $L_\cap$-definable equivalence relations on M. For all $\Box \in I \cup \{\cup, \cap\}$, let $L_\Box^{\cap-\mathrm{eq}}$ be the language obtained by adding these new sorts and function symbols to $L_\Box$ (note that we do not add quotients by L_i-definable equivalence relations), and let $\mathcal{M}_\Box^{\cap-\mathrm{eq}}$ be the natural expansion of $\mathcal{M}_\Box$ to $L_\Box^{\cap-\mathrm{eq}}$. Then $\mathcal{M}_\cup$ is interpolative if and only if $\mathcal{M}_\cup^{\cap-\mathrm{eq}}$ is interpolative. This follows from the fact that if $X_\Box$ is an $\mathcal{M}_\Box^{\cap-\mathrm{eq}}$-definable set in one of the new sorts, corresponding to the quotient of M^x by an $L_\cap$-definable equivalence relation, then the preimage of $X_\Box$ under the quotient is $\mathcal{M}_\Box$-definable.

We recall the classical Craig-Lyndon interpolation theorem (see, for example, [Hod93, Theorem 6.6.3]).

Theorem 3.2. *Suppose L_1 and L_2 are first order languages with intersection $L_\cap$ and φ_i is an L_i-sentence for $i \in \{1, 2\}$. If $\models (\varphi_1 \to \varphi_2)$ then there is an $L_\cap$-sentence ψ such that $\models (\varphi_1 \to \psi)$ and $\models (\psi \to \varphi_2)$. Equivalently: $\{\varphi_1, \varphi_2\}$ is inconsistent if and only if there is an $L_\cap$-sentence ψ such that $\models (\varphi_1 \to \psi)$ and $\models (\varphi_2 \to \neg\psi)$.*

We make extensive use of the following easy generalization of Theorem 3.2.

Corollary 3.3. *For each $i \in I$, let $\Sigma_i(x)$ be a set of L_i-formulas. If $\bigcup_{i\in I} \Sigma_i(x)$ is inconsistent, then there is a finite subset $J \subseteq I$ and an $L_\cap$-formula $\varphi^i(x)$ for each $i \in J$ such that:*

$$\Sigma_i(x) \models \varphi^i(x) \text{ for all } i \in J, \text{and } \{\varphi^i(x) \mid i \in J\} \text{ is inconsistent.}$$

Proof. Using the standard trick of introducing a new constant for each free variable, we reduce to the case when x is the empty tuple of variables. We may also assume that the sets $\Sigma_i(x)$ are closed under conjunction.

By compactness, if $\bigcup_{i\in I} \Sigma_i(x)$ is inconsistent, then there is a nonempty finite subset $J \subseteq I$ and a formula $\varphi_i(x) \in \Sigma_i(x)$ for all $i \in J$ such that $\{\varphi_i(x) \mid i \in J\}$ is inconsistent. For the sake of notational simplicity, we suppose $J = \{1, \ldots, n\}$. We argue by induction on n.

If $n = 1$, then we take φ^1 to be the contradictory $L_\cap$-formula $\bot$.

Suppose $n \geq 2$. Then $(\varphi_1 \wedge \ldots \wedge \varphi_{n-1})$ is an $(L_1 \cup \ldots \cup L_{n-1})$-sentence and

$$\{(\varphi_1 \wedge \ldots \wedge \varphi_{n-1}), \varphi_n\} \text{ is inconsistent.}$$

Applying Theorem 3.2, we get a sentence ψ in $L_n \cap (L_1 \cup \ldots \cup L_n) = L_\cap$ such that

$$\vDash (\varphi_1 \wedge \ldots \wedge \varphi_{n-1}) \to \psi \quad \text{and} \quad \vDash \varphi_n \to \neg\psi.$$

Then $\{\varphi_i \wedge \neg\psi \mid i \leq n-1\}$ is inconsistent and $\varphi_i \wedge \neg\psi$ is an L_i-sentence for $1 \leq i \leq n-1$. Applying induction, we choose for each $1 \leq i \leq n-1$ an L-sentence θ^i such that

$$\vDash (\varphi_i \wedge \neg\psi) \to \theta^i \text{ for all } 1 \leq i \leq n-1, \text{and } \vDash \neg(\theta^1 \wedge \ldots \wedge \theta^{n-1}).$$

Finally, set φ^i to be $(\psi \vee \theta^i)$ for $1 \leq i \leq n-1$ and φ^n to be $\neg\psi$. It is easy to check that all the desired conditions are satisfied. □

The following consistency condition for types follows immediately from Corollary 3.3. This is the generalization to our context of Robinson's joint consistency theorem.

Corollary 3.4. *Let $p(x)$ be a complete $L_\cap$-type, and for all $i \in I$, let $p_i(x)$ be a complete L_i-type such that $p(x) \subseteq p_i(x)$. Then $\bigcup_{i\in I} p_i(x)$ is consistent.*

The following lemma says that any family of definable sets which is not separated is "potentially" non-empty.

Lemma 3.5. *Let $\mathcal{M}_\cup$ be an $L_\cup$-structure, and suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$. The family $(X_i)_{i\in J}$ is separated if and only if for every $L_\cup$-structure $\mathcal{N}_\cup$ such that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, $\bigcap_{i\in J} X_i(\mathcal{N}_\cup) = \varnothing$.*

Proof. Suppose $(X_i)_{i\in J}$ is separated. Then there are $\mathcal{M}_\cap$-definable $X^1, \ldots, X^n$ such that $X_i \subseteq X^i$ for all $i \in J$ and $\bigcap_{i\in J} X^n = \varnothing$. Suppose $\mathcal{N}_\cup$ is a $T_\cup$-model satisfying $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$. Then $X_i(\mathcal{N}_\cup) \subseteq X^i(\mathcal{N}_\cup)$ for all $i \in J$ and $\bigcap_{i\in J} X^i(\mathcal{N}_\cup) = \varnothing$, so also $\bigcap_{i\in J} X_i(\mathcal{N}_\cup) = \varnothing$.

Conversely, suppose that for every $L_\cup$-structure $\mathcal{N}_\cup$ such that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, $\bigcap_{i\in J} X_i(\mathcal{N}_\cup) = \varnothing$. For each $i \in J$, let $\varphi_i(x, b)$ be an $L_i(M)$-formula defining X_i. Then the partial type

$$\bigcup_{i\in I} \mathrm{Ediag}(\mathcal{M}_i) \cup \bigcup_{i\in J} \varphi_i(x, b)$$

is inconsistent.

By compactness, there is a finite subset $J' \subseteq I$ with $J \subseteq J'$, a finite tuple $c \in M^y$ and a formula $\psi_i(b, c) \in \mathrm{Ediag}(\mathcal{M}_i)$ for each $i \in J'$ such that

$$\{\psi_i(b, c) \mid i \in J'\} \cup \{\varphi_i(x, b) \mid i \in J\}$$

is inconsistent. Let φ_i be the true formula $\top$ when $i \in J' \setminus J$, and define $\varphi'_i(x, y, z) = \varphi_i(x, y) \wedge \psi_i(y, z)$ for all $i \in J'$. Note that since $\mathcal{M}_i \vDash \psi_i(b, c)$, the formulas $\varphi_i(x, b)$ and $\varphi'_i(x, b, c)$ define the same subsets of M^x.

Applying Lemma 3.3, we obtain an an inconsistent family $\{\theta_i(x, y, z) \mid i \in J'\}$ of $L_\cap$-formulas such that $\vDash \varphi'_i(x, y, z) \to \theta_i(x, y, z)$ for each $i \in J'$. It follows that $\varphi_i(\mathcal{M}, b, c) \subseteq \theta_i(\mathcal{M}, b, c)$ for all $i \in J'$, and $\bigcap_{i\in J'} \theta_i(\mathcal{M}, b, c) = \varnothing$.

But since $\varphi_i(\mathcal{M}, b, c) = M^x$ when $i \in J' \setminus J$, also $\theta_i(\mathcal{M}, b, c) = M^x$ when $i \in J' \setminus J$. So already $\bigcap_{i\in J} \theta_i(\mathcal{M}, b, c) = \varnothing$, and the family $(\theta_i(\mathcal{M}, b, c))_{i\in J}$ separates $(X_i)_{i\in J}$. □

Lemma 3.5 can be used to characterize the interpolative $L_\cup$-structure and shows that every $L_\cup$-structure embeds in an interpolative $L_\cup$-structure in a way which is elementary for each L_i. Instead of proving this now, we will obtain this result as a consequence of a more general statement in the next section (see Proposition 3.8 below).

3.2. **Interpolative fusions.** Let T_i be a consistent L_i-theory for each $i \in I$, such that each T_i has the same set $T_\cap$ of $L_\cap$-consequences. Let $T_\cup = \bigcup_{i\in I} T_i$. Note the theories T_i, $T_\cup$, and $T_\cap$ may be incomplete.

The consistency of $T_\cup$ follows immediately from Corollary 3.3 and our assumption that the theories T_i have the same set of $T_\cap$-consequences. This assumption is not hard to arrange: given an arbitrary family of L_i-theories $(T_i)_{i\in I}$ we can extend each T_i with the set of all $L_\cap$-consequences of $\bigcup_{i\in I} T_i$.

We define the **interpolative fusion** $T^*_\cup$ of $(T_i)_{i\in I}$ over $T_\cap$ to be the theory of the interpolative $T_\cup$-models when the class of interpolative $T_\cup$-models is elementary. We say that "$T^*_\cup$ exists" if the class of interpolative $T_\cup$-models is elementary.

We will now show that when each T_i is model-complete, the interpolative $T_\cup$-models are exactly the existentially closed $T_\cup$-models, and the interpolative fusion $T^*_\cup$ is the model companion of $T_\cup$.

Remark 3.6. Any flat literal $L_\cup$-formula (see Section 2.1) is an L_i-formula for some $i \in I$. This trivial observation has two important consequences:

(1) If $\varphi(x)$ is a flat $L_\cup$-formula, then there is some finite $J \subseteq I$ and a flat L_i-formula $\varphi_i(x)$ for all $i \in J$ such that $\varphi(x)$ is logically equivalent to $\bigwedge_{i\in J} \varphi_i(x)$.
(2) If $\mathcal{A}_\cup$ is an $L_\cup$-structure, then $\mathrm{fdiag}(\mathcal{A}_\cup) = \bigcup_{i\in I} \mathrm{fdiag}_{L_i}(\mathcal{A}_i)$.

Theorem 3.7. *Suppose each T_i is model-complete. Then $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if it is existentially closed in the class of $T_\cup$-models. Hence, $T^*_\cup$ is precisely the model companion of $T_\cup$, if either of these exists.*

Proof. We prove the first statement under the given assumption. Suppose $\mathcal{M} \models T_\cup$ is existentially closed. Suppose $J \subseteq I$ is finite, and $\varphi_i(x)$ is an $L_i(M)$-formula for each $i \in J$ such that $(\varphi_i(M^x))_{i\in J}$ is not separated. We may assume each $\varphi_i(x)$ is existential as T_i is model-complete. Lemma 3.5 gives a $T_\cup$-model $\mathcal{N}$ extending $\mathcal{M}$ such that $\mathcal{N} \models \exists x \bigwedge_{i\in J} \varphi_i(x)$. As $\mathcal{M}$ is existentially closed and each φ_i is existential we have $\mathcal{M} \models \exists x \bigwedge_{i\in J} \varphi_i(x)$. Thus $\mathcal{M}$ is interpolative.

Now suppose $\mathcal{M} \models T_\cup$ is interpolative. Suppose $\psi(x)$ is a quantifier-free $L_\cup(M)$-formula and $\mathcal{N}$ is a $T_\cup$-model extending $\mathcal{M}$ such that $\mathcal{N} \models \exists x\, \psi(x)$. Applying Corollary 2.3, $\psi(x)$ is logically equivalent to a finite disjunction of E$\flat$-formulas $\bigvee_{k=1}^n \exists y_k\, \psi_k(x, y_k)$. Then for some k, $\mathcal{N} \models \exists x\, \exists y_k\, \psi_k(x, y_k)$. By Remark 3.6, the flat $L_\cup(M)$-formula $\psi_k(x, y_k)$ is equivalent to a conjunction $\bigwedge_{i\in J} \varphi_i(x, y_k)$ where $J \subseteq I$ is finite and $\varphi_i(x, y_k)$ is a flat $L_i(M)$-formula for each $i \in J$. So $\mathcal{N} \models \exists x\, \exists y_k \bigwedge_{i\in J} \varphi_i(x, y_k)$. As each T_i is model-complete, we have $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$. By Lemma 3.5, the sets defined by $\varphi_i(x, y_k)$ are not separated, and since $\mathcal{M}$ is interpolative, $\mathcal{M} \models \exists x \, \exists y_k \bigwedge_{i \in J} \varphi_i(x, y_k)$. So $\mathcal{M} \models \exists x \, \exists y_k \psi_k(x, y_k)$, and $\mathcal{M} \models \exists x \, \psi(x)$.

By Facts 2.6 and 2.7, each T_i has an axiomatization by $\forall\exists$-sentences, so $T_\cup$ does too. Hence $T_\cup$ is inductive and Fact 2.8 applies. The second statement then follows from the first statement. □

We have characterized the interpolative $T_\cup$ models in the case that each T_i is model-complete. We can reduce the general case to this one by Morleyizing (see [Hod93, Theorem 2.6.5]).

Proposition 3.8. *A model $\mathcal{M}_\cup$ of $T_\cup$ is interpolative if and only if for all finite $J \subseteq I$, family $(X_i)_{i \in J}$ of subsets of M^x such that X_i is $\mathcal{M}_i$-definable for all $i \in J$, and $T_\cup$-model $\mathcal{N}_\cup$ extending $\mathcal{M}_\cup$ with $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, we have that*

$$\bigcap_{i \in J} X_i(\mathcal{N}_\cup) \neq \varnothing \quad \textit{implies} \quad \bigcap_{i \in J} X_i \neq \varnothing.$$

Every $T_\cup$-model $\mathcal{M}_\cup$ can be embedded into an interpolative $T_\cup$-model $\mathcal{N}_\cup$ in such a way that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$.

Proof. Applying Remark 3.1(2,3), we can arrange that T_i admits quantifier-elimination, and L_i only consists of relational symbols for each $i \in I$. In particular, the condition $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ trivializes into the condition that $\mathcal{M}_i$ is an L_i-substructure of $\mathcal{N}_i$. The first statement is then an immediate consequence of Theorem 3.7. As T_i admits quantifier elimination, it is model complete and hence inductive. Hence, $T_\cup$ is also inductive. The second statement is then a consequence of Theorem 3.7 and Fact 2.8, which states that every model of an inductive theory can be embedded into an existentially closed model of that theory. □

4. Preservation results

Throughout this section, we use the notational conventions of Section 3, fixing languages $L_\square$ and theories $T_\square$ for $\square \in I \cup \{\cup, \cap\}$ and we assume the interpolative fusion $T^*_\cup$ exists.

Many of the results in this section contain the hypothesis "suppose $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i". When $T_\cap$ or T_i is incomplete, we mean that this property holds in any consistent completions of these theories. By Proposition 2.24, this hypothesis is always satisfied by $\mathop{\smile\!\!\!\!\!\!|}^f$ when $T_\cap$ is stable with weak elimination of imaginaries. For example, this applies when $T_\cap$ is the theory of an infinite set or the theory of algebraically closed fields. In the general case, elimination of imaginaries for $T_\cap$ is easily arranged (see Remark 3.1).

Remark 4.1. Much of this section is devoted to $\mathcal{K}$-completeness of $T^*_\cup$ for various classes $\mathcal{K}$ (see Section 2.2). By Remarks 2.4 and 3.6, if $T^*_\cup$ is $\mathcal{K}$-complete, then for any structure $\mathcal{A}_\cup$ in $\mathcal{K}$ such that $\mathcal{A}_\cup \subseteq \mathcal{M}_\cup \models T^*_\cup$,

$$T^*_\cup \cup \bigcup_{i \in I} \mathrm{fdiag}_{L_i}(\mathcal{A}_i) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}_\cup).$$

4.1. **Preservation of model-completeness.** We interpret Theorem 3.7 as a first preservation result.

Theorem 4.2. *Suppose each T_i is model-complete. Then $T^*_\cup$ is model-complete, and every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*

$$\exists y \bigwedge_{i \in J} \varphi_i(x, y),$$

where $J \subseteq I$ is finite and each $\varphi_i(x, y)$ is a flat L_i-formula.

Proof. The first assertion follows immediately from Theorem 3.7. Since $T^*_\cup$ is model-complete, $\psi(x)$ is $T^*_\cup$-equivalent to an existential $L_\cup$-formula $\exists z\, \varphi(x, z)$. By Corollary 2.3, $\varphi(x, z)$ is equivalent to a finite disjunction of E$\flat$-formulas. Distributing the quantifier $\exists z$ over the disjunction and applying Remark 3.6 yields the desired result. □

4.2. **Preservation of** acl**- and** bcl**-completeness.** Given $\Box \in I \cup \{\cup, \cap\}$, let $\mathrm{acl}_\Box(A)$ be the $\mathcal{M}_\Box$-algebraic closure of a subset A of a $T^*_\cup$-model $\mathcal{M}_\cup$. The **combined closure**, $\mathrm{ccl}(A)$, of a subset A of $\mathcal{M}_\cup$ is the smallest set containing A which is acl_i-closed for each $i \in I$. More concretely, $b \in \mathrm{ccl}(A)$ if and only if

$$b \in \mathrm{acl}_{i_n}(\dots(\mathrm{acl}_{i_1}(A))\dots) \text{ for some } i_1, \dots, i_n \in I.$$

Theorem 4.3. *Suppose $T_\cap$ admits a stationary independence relation $\mathop{\smile\!\!\!\!\!\!|}$ which satisfies full extension in T_i for all i. If each T_i is* acl*-complete then $T^*_\cup$ is* acl*-complete and $\mathrm{acl}_\cup = \mathrm{ccl}$.*

Proof. Theorem 4.2 shows $T^*_\cup$ is model complete. In order to apply Theorem 2.5, we will show that the class of $T^*_\cup$-models has the disjoint amalgamation property over ccl-closed substructures.

So suppose $\mathcal{A}_\cup$ is a ccl-closed substructure of a $T^*_\cup$-model $\mathcal{M}_\cup$ and $f : \mathcal{A}_\cup \to \mathcal{N}_\cup \models T^*_\cup$ is an embedding. Let $\boldsymbol{\mathcal{M}}_\cup$ be a monster model of $\widehat{T}_\cup = \mathrm{Th}_{L_\cup}(\mathcal{N}_\cup)$, so $\mathcal{N}_\cup$ is an elementary substructure of $\boldsymbol{\mathcal{M}}_\cup$. Let $A' = f(A) \subseteq N$. Let $p_\Box(x) = \mathrm{tp}_{L_\Box}(M/A)$ for each $\Box \in I \cup \{\cap\}$, where x is a tuple of variables enumerating M. By acl-completeness of T_i, $f : \mathcal{A}_i \to \mathcal{N}_i$ is partial elementary for all $i \in I$, so $f : \mathcal{A}_\cap \to \mathcal{N}_\cap$ is also partial elementary, and we can replace the parameters from A in $p_\Box(x)$ by their images under f, obtaining a consistent type $p'_\Box(x)$ over A' for all $\Box \in I \cup \{\cap\}$.

Fix $i \in I$. Since A is algebraically closed in $\mathcal{M}_i$, A' is algebraically closed in $\boldsymbol{\mathcal{M}}_i$. By full existence for $\mathop{\smile\!\!\!\!\!\!|}$ in T_i, there is a realization M'_i of $p'_i(x)$ in $\boldsymbol{\mathcal{M}}_i$ such that $M'_i \mathop{\smile\!\!\!\!\!\!|}_{A'} N$ in $\boldsymbol{\mathcal{M}}_i$. Let $q_i(x) = \mathrm{tp}_{L_i}(M'_i/N)$.

For all $i, j \in I$, $\mathrm{tp}_{L_\cap}(M'_i/A') = \mathrm{tp}_{L_\cap}(M'_j/A') = p'_\cap(x)$, so by stationarity for $\mathop{\smile\!\!\!\!\!\!|}$, $\mathrm{tp}_{L_\cap}(M'_i/N) = \mathrm{tp}_{L_\cap}(M'_j/N)$. Let $q_\cap(x)$ be this common type, so $q_\cap(x) \subseteq q_i(x)$ for all i. We claim that $\bigcup_{i \in I} q_i(x)$ is realized an an elementary extension of $\mathcal{N}_\cup$.

By Lemma 3.4, the partial $L_\cup(N)$-type

$$\bigcup_{i \in I} (\mathrm{Ediag}(\mathcal{N}_i) \cup q_i(x))$$

is consistent, since each $L_i(N)$-type $(\mathrm{Ediag}(\mathcal{N}_i) \cup q_i(x))$ contains the complete $L_\cap(N)$-type $(\mathrm{Ediag}(\mathcal{N}_\cap) \cup q_\cap(x))$. Suppose it is realized by M'' in $\mathcal{N}'_\cup$. Then M'' is the domain of a substructure $\mathcal{M}''_\cup$ isomorphic to $\mathcal{M}_\cup$ via the enumeration of both structures by the variables x. Let $f' : \mathcal{M}_\cup \to \mathcal{M}''_\cup$ be this isomorphism. Also $\mathcal{N}_i \preceq \mathcal{N}'_i$ for all $i \in I$, and in particular $\mathcal{N}'_\cup \models T_\cup$. Since $T_\cup$ is inductive, there is an extension $\mathcal{N}^*_\cup$ of $\mathcal{N}'_\cup$ such that $\mathcal{N}^*_\cup$ is existentially closed, i.e., $\mathcal{N}^*_\cup \models T^*_\cup$. Since each T_i is model-complete, we have $\mathcal{N}'_i \preceq \mathcal{N}^*_i$ for all $i \in I$, so M'' satisfies $\bigcup_{i \in I} q_i(x)$ in $\mathcal{N}^*_\cup$. And since $T^*_\cup$ is model-complete, $\mathcal{N}_\cup \preceq \mathcal{N}^*_\cup$.

We view $\mathcal{N}^*_\cup$ as an elementary substructure of $\boldsymbol{\mathcal{M}}_\cup$, and we view f' as an embedding $\mathcal{M}_\cup \to \mathcal{N}^*_\cup$. If $a \in A$, then a is enumerated by a variable x_a from x, and the formula $x_a = a$ is in $p_\cap(x)$. So $f'(a)$ satisfies the formula $x_a = f(a)$. This establishes the amalgamation property.

For the disjoint amalgamation property, note that we have $M'' \perp_{A'} N$ in $\boldsymbol{\mathcal{M}}_\cap$, so by algebraic independence for $\perp$, $M'' \cap N = A'$, and hence $f'(M) \cap N = f(A)$.

By Theorem 2.5, $T^*_\cup$ is ccl-complete and every ccl-closed substructure is $\mathrm{acl}_\cup$-closed. It follows that for any set $B \subseteq \mathcal{M} \models T$, $\mathrm{acl}_\cup(B) \subseteq \mathrm{ccl}(B)$.

For the converse, it suffices to show $\mathrm{acl}_\cup(B)$ is acl_i-closed for all $i \in I$. Indeed,

$$\mathrm{acl}_i(\mathrm{acl}_\cup(B)) \subseteq \mathrm{acl}_\cup(\mathrm{acl}_\cup(B)) = \mathrm{acl}_\cup(B).$$

So $\mathrm{acl}_\cup = \mathrm{ccl}$, and $T^*_\cup$ is acl-complete. □

Corollary 4.4. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Suppose each T_i is* bcl*-complete. Then $T^*_\cup$ is* bcl *complete and every $L_\cup$-formula is $T^*_\cup$-equivalent to a finite disjunction of b.e. formulas of the form*

$$\exists y \bigwedge_{i \in J} \varphi_i(x,y),$$

where $J \subseteq I$ is finite and $\varphi_i(x,y)$ is a flat L_i formula for all $i \in J$.

Proof. Theorem 2.13 implies T_i is acl-complete and $\mathrm{bcl}_i = \mathrm{acl}_i$ for all $i \in I$. We have $\mathrm{bcl}_\cup(A) \subseteq \mathrm{acl}_\cup(A)$ for any subset A of a $T_\cup$-model. But also, for all $i \in I$,

$$\begin{aligned}\mathrm{acl}_i(\mathrm{bcl}_\cup(A)) &= \mathrm{bcl}_i(\mathrm{bcl}_\cup(A))\\ &\subseteq \mathrm{bcl}_\cup(\mathrm{bcl}_\cup(A))\\ &= \mathrm{bcl}_\cup(A).\end{aligned}$$

So $\mathrm{bcl}_\cup(A)$ is acl_i-closed for all $i \in I$, hence $\mathrm{ccl}(A) \subseteq \mathrm{bcl}_\cup(A)$.

Theorem 4.3 implies $T^*_\cup$ is acl-complete and $\mathrm{ccl}(A) = \mathrm{bcl}_\cup(A) = \mathrm{acl}_\cup(A)$. Applying Theorem 2.13 again, $T^*_\cup$ is bcl-complete.

It remains to characterize $L_\cup$-formulas up to equivalence. Theorem 2.13 shows every $L_\cup$-formula is $T^*_\cup$-equivalent to a finite disjunction of b.e. formulas. Let $\exists y\, \psi(x,y)$ be a b.e. formula appearing in the disjunction. By Corollary 2.3, the quantifier-free formula $\psi(x,y)$ is equivalent to a finite disjunction of $\mathrm{E}\flat$-formulas $\bigvee_{j=1}^m \exists z_j\, \theta_j(x,y,z_j)$. Distributing the quantifier $\exists y$ over the disjunction, we find that $\exists y \exists z_j\, \theta_j(x,y,z_j)$ is a b.e. formula. Applying Remark 3.6 to the flat formula $\theta_j(x,y,z_j)$ yields the result. □

We conclude with two counterexamples showing that the hypotheses on $T_\cap$ are necessary for acl-completeness of interpolative fusions. In the first example $T_\cap$ is unstable with elimination of imaginaries, the second example is stable but fails weak elimination of imaginaries. In neither example does $T_\cap$ admit a stationary independence relation which satisfies full existence in T_i for all i.

Example 4.5. Let $L_\cap = \{\leq\}$ and L_i be the expansion of $L_\cap$ by a unary predicate P_i for $i \in \{1,2\}$. Let $T_\cap = \mathrm{DLO}$, and T_i be the theory of a dense linear order equipped with a downwards closed supremum-less set defined by P_i for $i \in \{1,2\}$. Then T_1, T_2 admit quantifier elimination and $T_\cup$ has exactly two completions. A $L_\cup$-structure $\mathcal{M}_\cup$ is a $T^*_\cup$-model if and only if we either have $P_1(M) \subsetneq P_2(M)$ or $P_2(M) \subsetneq P_1(M)$. In either kind of model $\varnothing$ is easily seen to be algebraically closed. The completions of $T^*_\cup$ are not determined by $\mathrm{fdiag}_{L_1}(\varnothing) \cup \mathrm{fdiag}_{L_2}(\varnothing)$, so $T^*_\cup$ is not acl-complete.

Example 4.6. Let $L_\cap = \{E\}$ where E is a binary relation symbol. Let $L_i = \{E, P_i\}$ where P_i is unary for $i \in \{1,2\}$. Let $T_\cap$ be the theory of an equivalence relation with infinitely many infinite classes. Let T_i be the theory of a $T_\cap$-model with a distinguished equivalence class named by P_i. Note T_1, T_2 have quantifier elimination. A $T^*_\cup$-model $\mathcal{M}_\cup$ may have $P_1(M) = P_2(M)$ or $P_1(M) \neq P_2(M)$. Thus $T^*_\cup$ has two completions. Again, $\mathrm{acl}(\varnothing) = \varnothing$ and the completions are not determined by $\mathrm{fdiag}_{L_1}(\varnothing) \cup \mathrm{fdiag}_{L_2}(\varnothing)$, so $T^*_\cup$ is not acl-complete.

4.3. **Preservation of quantifier elimination.** When is quantifier elimination is preserved in interpolative fusions? In contrast to preservation of model-completeness, acl-completeness, and bcl-completeness, we cannot obtain quantifier elimination in $T^*_\cup$ without tight control on algebraic closure in the T_i. In this section we assume each T_i admits quantifier elimination, hence T_i is bcl-complete and $\mathrm{bcl}_i = \mathrm{acl}_i$ for all $i \in I$ by Theorem 2.13.

Theorem 4.7 below is motivated by some comments in the introduction of [MS14] on the failure of quantifier elimination in ACFA.

Theorem 4.7. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Suppose every T_i has quantifier elimination, and*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all $L_\cup$-substructures A of $T^*_\cup$-models $\mathcal{M}_\cup$ and all $i \in I$. Then $T^*_\cup$ has quantifier elimination.

Proof. Theorem 4.3 shows $T^*_\cup$ is ccl-complete. We show $T^*_\cup$ is substructure complete. Suppose $\mathcal{A}_\cup$ is an $L_\cup$-substructure of a $T^*_\cup$-model $\mathcal{M}_\cup$ and $f\colon \mathcal{A}_\cup \to \mathcal{N}_\cup \models T^*_\cup$ is an embedding.

Any $\mathrm{acl}_\cap$-closed subset of M is acl_i-closed for all $i \in I$. Hence,

$$\mathrm{acl}_\cap(A) = \mathrm{acl}_i(A) = \mathrm{ccl}(A)$$

for all $i \in I$. As each T_i is substructure complete, f is partial elementary $\mathcal{M}_i \to \mathcal{N}_i$, so f extends to a partial elementary map $g_i\colon \mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \to \mathcal{N}_i$.

Fix $j \in I$. For all $i \in I$, $g_i^{-1} \circ g_j$ is an $L_\cap$-automorphism of $\mathrm{acl}_\cap(A)$ fixing A pointwise, so in fact it is an L_i-automorphism. It follows that $g_j = g_i \circ (g_i^{-1} \circ g_j)$ is an L_i-embedding $\mathrm{acl}_\cap(A) \to \mathcal{N}_i$. Since i was arbitrary, g_j is an $L_\cup$-embedding.

But $\mathrm{acl}_\cap(A) = \mathrm{ccl}(A)$, so by ccl-completeness, g_j is partial elementary $\mathcal{M}_\cup \to \mathcal{N}_\cup$, and hence so is $g_j|_A = f$. □

We prefer hypothesis which can be checked language-by-language, i.e., which refer only to properties of T_i, $T_\cap$, and the relationship between T_i and $T_\cap$ rather than how T_i and T_j relate when $i \neq j$, or how T_i relates to $T_\cup$. The hypothesis of Theorem 4.7 is not strictly language-by-language, because it refers to an arbitrary $L_\cup$-substructure A. However, there are several natural strengthenings of this hypothesis which are language-by-language. One is to simply assume the hypothesis of Theorem 4.7 for all sets A. Two more are given in the following corollary.

Corollary 4.8. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Suppose each T_i admits quantifier elimination. If either of the following conditions hold for all sets A, then $T^*_\cup$ has quantifier elimination:*

(1) $\mathrm{acl}_i(A) = \langle A\rangle_{L_i}$ *for all* $i \in I$.
(2) $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ *for all* $i \in I$.

Proof. We apply Theorem 4.7, so assume $A = \langle A\rangle_{L_\cup}$.

(1) We have $A \subseteq \mathrm{acl}_\cap(A) \subseteq \mathrm{acl}_i(A) = \langle A\rangle_{L_i} = A$.
(2) We have $\mathrm{dcl}_\cap(A) \subseteq \mathrm{acl}_\cap(A) \subseteq \mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$.

In both cases, the group $\mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A)$ is already trivial, so its subgroup $\mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$ is too. □

Corollary 4.9. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Suppose each T_i admits quantifier elimination and a universal axiomization. Then $T^*_\cup$ has quantifier elimination.*

Proof. Every L_i-substructure of T_i is an elementary substructure, and hence acl_i-closed, so we can apply Corollary 4.8(1). □

4.4. **Consequences for general interpolative structures.** Many of the results above can be translated to the general case (when the T_i are not model-complete) by Morleyization (as in the proof of Proposition 3.8). This allows us to understand $L_\cup$-formulas and complete $L_\cup$-types relative to L_i-formulas and complete L_i-types.

To set notation: For each i, the Morleyization gives a definitional expansion $L_i^\diamond$ of L_i, and an extension $T_i^\diamond$ of T_i by axioms defining the new symbols in $L_i^\diamond$. We assume that the new symbols in $L_i^\diamond$ and $L_j^\diamond$ are distinct for $i \neq j$, so that $L_i^\diamond \cap L_j^\diamond = L_\cap$. It follows that each $T_i^\diamond$ has the same set of $L_\cap$ consequences, namely $T_\cap$. We let $L_\cup^\diamond = \bigcup_{i\in I} L_i^\diamond$ and $T_\cup^\diamond = \bigcup_{i\in I} T_i^\diamond$. Then every $T_\cup$-model $\mathcal{M}_\cup$ has a canonical expansion to a $T_\cup^\diamond$-model $\mathcal{M}_\cup^\diamond$, and by Remark 3.1, $\mathcal{M}_\cup$ is interpolative if and only if $\mathcal{M}_\cup^\diamond$ is interpolative. Since each $T_i^\diamond$ is model-complete, it follows that the interpolative fusion $T^*_\cup$ exists if and only if $T_\cup^\diamond$ has a model companion.

Proposition 4.10. *(1) Every formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*

$$\exists y \bigwedge_{i\in J} \varphi_i(x,y)$$

where $J \subseteq I$ is finite and $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$.

(2) If $\mathcal{M}_\cup$ is an $T^*_\cup$-model, then

$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{tp}_{L_i}(M) \models \mathrm{tp}_{L_\cup}(M).$$

Proof. For (1), each Morleyized theory $T_i^\diamond$ has quantifier elimination, hence is model-complete, so we can apply Theorem 4.2 to the theory $(T_\cup^\diamond)^*$ of interpolative $T_\cup^\diamond$ models. This says $(T_\cup^\diamond)^*$ is model-complete, and $\psi(x)$ is $(T_\cup^\diamond)^*$-equivalent to a finite disjunction of formulas of the form $\exists y \bigwedge_{i\in J} \varphi_i(x,y)$, where each $\varphi_i(x,y)$ is a flat $L_i^\diamond$-formula. But since $L_i^\diamond$ is a definitional expansion of L_i, each formula $\varphi_i(x,y)$ can be translated back to an L_i-formula.

For (2), since $(T_\cup^\diamond)^*$ is model-complete, we have

$$(T^*_\cup)^\diamond \cup \mathrm{fdiag}_{L_\cup^\diamond}(M) \models \mathrm{tp}_{L_\cup^\diamond}(M).$$

But $\mathrm{fdiag}_{L_\cup^\diamond}(M) = \bigcup_{i\in I} \mathrm{fdiag}_{L_i^\diamond}(M)$, and $\mathrm{fdiag}_{L_i^\diamond}(M)$ is completely determined by $\mathrm{tp}_{L_i}(M)$, so the result follows. □

Proposition 4.10(2) can be rephrased as a kind of relative model-completeness: If $\mathcal{M}_\cup$ and $\mathcal{N}_\cup$ are $T^*_\cup$-models with $M \subseteq N$ and $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$, then $\mathcal{M}_\cup \preceq \mathcal{N}_\cup$.

We will now establish a sequence of variants on Proposition 4.10, with stronger hypotheses and stronger conclusions.

Proposition 4.11. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i.*

1. *Every formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas*
$$\exists y \bigwedge_{i\in J} \varphi_i(x,y)$$
where $J \subseteq I$ is finite, $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i\in J}\varphi_i(x,y)$ is bounded in y.
2. *If A is an algebraically closed subset of a $T^*_\cup$-model $\mathcal{M}$, then*
$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{tp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

Proof. Just as in the proof of Proposition 4.10, but this time using the fact that $(T^\diamond_\cup)^*$ is bcl-complete and applying Corollary 4.4. □

It follows from Proposition 4.11 that if $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i, then the completions of $T^*_\cup$ are determined by the L_i-types of $\mathrm{acl}_\cup(\varnothing)$ for all i.

Proposition 4.12. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Suppose further that $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A)$ and $\mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$ for all $L_\cup$-substructures A and all $i \in I$. Then:*

1. *Every formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas*
$$\exists y \bigwedge_{i\in J} \varphi_i(x,y)$$
where $J \subseteq I$ is finite, $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i\in J}\varphi_i(x,y)$ is bounded in y with bound 1.
2. *If A is an $L_\cup$-substructure of a $T^*_\cup$-model $\mathcal{M}$ then*
$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{tp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

Proof. Observing that Morleyization does not affect our hypotheses about acl_i and $\mathrm{acl}_\cap$, we find that $(T^\diamond_\cup)^*$ has quantifier elimination, by Theorem 4.7. This gives us (2) as in the proof of Proposition 4.10.

For (1), note that $\psi(x)$ is $(T^\diamond_\cup)^*$-equivalent to a quantifier-free formula. The result then follows from Corollary 2.3 and Remark 3.6. □

As in Corollary 4.8, we can replace the hypotheses of Proposition 4.12 with: $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i, and either of the following stronger conditions:

1. $\mathrm{acl}_i(A) = A$ for all $L_\cup$-substructures A and all $i \in I$.
2. $\mathrm{acl}_i(A) = \langle A\rangle_i$ for all sets A and all $i \in I$.

We have to assume something slightly stronger if we want to get true quantifier elimination down to L_i-formulas in $T^*_\cup$.

Proposition 4.13. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Suppose further that $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A)$ and $\mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$ for all sets A and all $i \in I$. Then:*

(1) *Every formula is $T^*_\cup$-equivalent to a Boolean combination of L_i-formulas.*
(2) *If A is any set, then*

$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{tp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

Proof. We first move to a relational language by replacing all function symbols by their graphs. Then we proceed just as in the proof of Proposition 4.12, noting that when $L^\diamond_\cup$ is relational, a quantifier-free $L^\diamond_\cup$-formula is already a Boolean combination of $L^\diamond_i$-formulas. □

Once again, we can replace the hypotheses of Proposition 4.13 with a stronger, but somewhat more convenient, condition: $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i, and $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ for all sets A.

4.5. **Preservation of stability and NIP.** In this subsection we give applications of some of the technical work above.

Proposition 4.14. *Assume the hypotheses of Proposition 4.13. If each T_i is stable (NIP), then $T^*_\cup$ is stable (NIP).*

Proof. This follows immediately from Proposition 4.13(1) as stable (NIP) formulas are closed under Boolean combinations. □

We can also use Proposition 4.13 to count types.

Proposition 4.15. *Assume the hypotheses of Proposition 4.13, and suppose that I is finite. If each T_i is stable in κ, then $T^*_\cup$ is stable in κ. As a consequence, if each T_i is $\aleph_0$-stable then $T^*_\cup$ is $\aleph_0$-stable, and if each T_i is superstable, then $T^*_\cup$ is superstable.*

Proof. We consider $S_x(A)$, where x is a finite tuple of variables, $A \subseteq \mathcal{M} \models T^*_\cup$, and $|A| \leq \kappa$. By Proposition 4.13(2), a type in $S_x(A)$ is completely determined by its restrictions to L_i for all $i \in I$. Since there are at most κ L_i-types over A in the variables x, we have $|S_x(A)| \leq \prod_{i\in I} \kappa = \kappa$, since I is finite. □

We do not expect to obtain preservation of stability or NIP without strong restrictions on acl, as in the hypotheses of Proposition 4.13. The proofs of Propositions 4.14 and 4.15 do not apply to other classification-theoretic properties, like simplicity, NSOP_1, NTP_2, etc., as these properties are not characterized by counting types, and formulas with these properties are not closed under Boolean combinations in general. However, we can obtain preservation results for some of these properties under more general hypotheses. These results will be contained in the next paper [KTW].

4.6. **Preservation of $\aleph_0$-categoricity.** In this subsection, we do not assume that the interpolative fusion $T^*_\cup$ exists. Applying the preservation results above, we show that $T^*_\cup$ exists and is $\aleph_0$-categorical provided that certain hypotheses, including $\aleph_0$-categoricity, on the T_i hold. This section is closely related to work of Pillay and Tsuboi [PT97].

Proposition 4.16. *Assume $T_\cap$ admits a stationary independence relation which satisfies full extension in T_i for all i. Assume also that all languages have only finitely many sorts. Suppose that each T_i is $\aleph_0$-categorical and that there is some $i^* \in I$ such that $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A)$ for all $i \neq i^*$. Then the class of interpolative $T_\cup$-models is elementary.*

Proof. A $T_\cup$-model $\mathcal{M}_\cup$ has the **joint consistency property** if for every finite $B \subseteq M$ such that $B = \mathrm{acl}_{i^*}(B)$ and every family $(p_i(x))_{i\in J}$ such that J is a finite subset of I, $p_i(x)$ is a complete L_i-type over B for all $i \in J$, and the p_i have a common restriction $p_\cap(x)$ to $L_\cap$, then $\bigcup_{i\in I} p_i(x)$ is realized in $\mathcal{M}_\cup$.

Note that the joint consistency property is elementary. Indeed, by $\aleph_0$-categoricity, there is an L_{i^*}-formula $\psi(y)$ expressing the property that the set B enumerated by a tuple b is acl_{i^*}-closed. Since B is finite, every complete type $p_i(x)$ over B is isolated by a single formula. And the property that the L_i-formula $\varphi_i(x,b)$ isolates a complete L_i-type over B whose restriction to $L_\cap$ is isolated by the $L_\cap$-formula $\varphi_\cap(x,b)$ is definable by a formula $\theta_{\varphi_i,\varphi_\cap}(b)$. So the class of $T_\cup$-models with the joint consistency property is axiomatized by $T_\cup$ together with sentences of the form

$$\forall y \left[\left(\psi(y) \wedge \bigwedge_{i\in J} \theta_{\varphi_i,\varphi_\cap}(y)\right) \to \exists x \bigwedge_{i\in J} \varphi_i(x,y)\right].$$

It remains to show that a structure $\mathcal{M}_\cup$ is interpolative if and only if it has the joint consistency property. So suppose $\mathcal{M}_\cup$ is interpolative, let B and $(p_i(x))_{i\in J}$ be as in the definition of the joint consistency property, and suppose for contradiction that $\bigcup_{i\in J} p_i(x)$ is not realized in $\mathcal{M}_\cup$. Note that since B is acl_{i^*}-closed, it is also acl_i-closed for all $i \neq i^*$, since $\mathrm{acl}_i(B) = \mathrm{acl}_\cap(B) \subseteq \mathrm{acl}_{i^*}(B) = B$.

Each $p_i(x)$ is isolated by a single L_i-formula $\varphi_i(x,b)$, and

$$\mathcal{M}_\cup \models \neg\exists x \bigwedge_{i\in J} \varphi_i(x,b).$$

It follows that the φ_i are separated by a family of $L_\cap$-formulas $(\psi^i(x,c_i))_{i\in J}$. Let $C = B \cup \{c_i \mid i \in J\}$. By full existence for $\mathop{\smile\!\!\!\!\!\!|}$ in T_i, since B is acl_i-closed, $p_i(x)$ has an extension to a type $q_i(x)$ over C such that for any realization a_i of $q_i(x)$, $a \mathop{\smile\!\!\!\!\!\!|}_B C$. By stationarity, the types $q_i(x)$ have a common restriction $q_\cap$ to $L_\cap$. Now for all $i \in J$, since $\varphi_i(x,b) \in p_i(x)$, $\psi^i(x,c_i) \in q_i(x)$, and hence $\psi^i(x,c_i) \in q_\cap(x)$. This is a contradiction, since $\{\psi^i(x,c_i) \mid i \in J\}$ is inconsistent.

Conversely, suppose $\mathcal{M}_\cup$ has the joint consistency property. Let $(\varphi_i(x,a_i))_{i\in J}$ be a family of formulas which are not separated. Let $B = \mathrm{acl}_{i^*}((a_i)_{i\in J})$. Since T_{i^*} is $\aleph_0$-categorical, and there are only finitely many sorts, B is finite. For each $i \in J$, there is an $L_\cap$-formula $\psi^i(x,b)$ such that $\mathcal{M}_\cup \models \psi^i(a,b)$ if and only if $\mathrm{tp}_{L_\cap}(a/B)$ is consistent with $\varphi_i(x,a_i)$ (indeed, $\psi^i(x,b)$ is the disjunction of formulas isolating each of the finitely many such types). Since the formulas $\psi^i(x,b)$ do not separate the formulas $\varphi_i(x,a_i)$, there must be some element $a \in M^x$ satisfying $\bigwedge_{i\in J} \psi^i(x,b)$. Then $p_\cap(x) = \mathrm{tp}_{L_\cap}(a/B)$ is consistent with each $\varphi_i(x,a_i)$, so $p_\cap(x) \cup \{\varphi_i(x,a_i)\}$ can be extended to a complete L_i-type $p_i(x)$ over B. By the joint consistency property, there is some element in M^x realizing $\bigcup_{i\in J} p_i(x)$, and in particular satisfying $\bigwedge_{i\in J} \varphi_i(x,a_i)$. □

A type-counting argument as in Proposition 4.15 now gives preservation of $\aleph_0$-categoricity.

Theorem 4.17. *Assume the hypotheses of Proposition 4.16, and let $T^*_\cup$ be the interpolative fusion. Assume additionally that I is finite. Then every completion of $T^*_\cup$ is $\aleph_0$-categorical.*

Proof. Let $\widehat{T}$ be a completion of $T^*_\cup$. It suffices to show that for any finite tuple of variables x, there are only finitely many $L_\cup$-types over the empty set in the variables x relative to $\widehat{T}$. Since $\mathrm{acl}_\cup = \mathrm{acl}_{i^*}$ is uniformly locally finite, there is an upper bound m on the size of $\mathrm{acl}_\cup(a)$ for any tuple $a \in M^x$ when $M \models \widehat{T}$.

By Proposition 4.11, $\mathrm{tp}_{L_\cup}(\mathrm{acl}_\cup(a))$ is determined by $\bigcup_{i\in I} \mathrm{tp}_{L_i}(\mathrm{acl}_\cup(a))$. So the number of possible $L_\cup$-types of a is bounded above by the product over all i of the number of L_i-types of m-tuples relative to T_i. This is finite, since I is finite and each T_i is $\aleph_0$-categorical. □

We recover the following result of Pillay and Tsuboi.

Corollary 4.18 ([PT97, Corollary 5]). *Assume $T_\cap$ is stable with weak elimination of imaginaries. Let $I = \{1, 2\}$, suppose T_1 and T_2 are $\aleph_0$-categorical single-sorted theories, and suppose $\mathrm{acl}_1(A) = \mathrm{acl}_\cap(A)$ for all $A \subseteq \boldsymbol{M}_1$. Then $T_\cup$ admits an $\aleph_0$-categorical completion.*

5. Existence results

Throughout this section, we assume in addition to the settings of section 3 that $L \subseteq L'$ are first-order languages, $\mathcal{M}$ and $\mathcal{M}'$ are an L-structures and an L'-structures both with underlying collection of sorts M, T' is an L'-theory, and T is the set of L-consequences of T'.

The goal of the section is to provide sufficient conditions for the existence of the interpolative fusion. For this purpose, it is useful to find simpler characterizations of interpolative $T_\cup$-models. In Sections 5.1, we accomplish this in the setting when $T_\cap$ admits an ordinal-valued dimension function, highlighting the notions of approximability and definability of pseudo-denseness in the expansions T_i. In Section 5.2, we show how to relativize these conditions to collections of definable sets we call pseudo-cells. In the remaining sections, we investigate these notions under additional hypotheses on $T_\cap$ (e.g. $\aleph_0$-stability, o-minimality).

5.1. The pseudo-topological axioms.

In an interpolative structure $\mathcal{M}_\cup$, any finite family of inseparable L_i-definable sets must have non-empty intersection. Heuristically, if each X_i is "large" in a some fixed $X_\cap$, then $(X_i)_{i\in I}$ is inseparable. When $T_\cap$ has a reasonable notion of dimension we can make this idea precise. The setting has a certain topological flavor, hence the name.

Throughout this section, we assume the existence of a function dim, which assigns an ordinal or the formal symbol $-\infty$ to each $\mathcal{M}$-definable set so that for all $\mathcal{M}$-definable $X, X' \subseteq M^x$:

(1) $\dim(X \cup X') = \max\{\dim X, \dim X'\}$,
(2) $\dim X = -\infty$ if and only if $X = \varnothing$,
(3) $\dim X = 0$ if and only if X is nonempty and finite,

We call such a function dim an **ordinal rank** on $\mathcal{M}$. A function on the collection of definable sets in T-models that restricts to an ordinal rank on each T-model is called an **ordinal rank** on T. If dim has the extra property that $\dim X(\mathcal{M}) = \dim X(\mathcal{N})$

for all $\mathcal{M} \models T$, $\mathcal{M}$-definable sets X, and elementary extensions $\mathcal{N}$ of $\mathcal{M}$, we call dim an **elementary rank** on T.

We can equip any theory with a trivial elementary rank by declaring $\dim(X) = 1$ whenever X is infinite. Tame theories are generally equipped with a natural (often canonical) elementary rank. Examples are $\aleph_0$-stable theories with Morley rank, superstable theories with U-rank, and supersimple theories with SU-rank.

Let X be a definable subset of M^x and A be an arbitrary subset of M^x. Then A is **pseudo-dense** in X if A intersects every non-empty definable $X' \subseteq X$ such that $\dim X' = \dim X$. We call X a **pseudo-closure** of A if $A \subseteq X$ and A is pseudo-dense in X. The following lemma collects a few easy facts about pseudo-denseness, the proofs of which we leave to the readers.

Lemma 5.1. *Let X and X' be $\mathcal{M}$-definable subsets of M^x, and let A be an arbitrary subset of M^x. Then:*

1. *When X is finite, A is pseudo-dense in X if and only if $X \subseteq A$.*
2. *If A is pseudo-dense in X, $X' \subseteq X$, and $\dim X' = \dim X$, then A is pseudo-dense in X'.*
3. *If $X^1, \dots, X^n \subseteq X$ are $\mathcal{M}$-definable, with $\dim X^i = \dim X$ for all i, and*
$$\dim X \triangle (X^1 \cup \dots \cup X^n) < \dim X,$$
then A is pseudo-dense in X if and only if A is pseudo-dense in each X^i.

If in addition X is a pseudo-closure of A, then:

4. *$A \subseteq X'$ implies $\dim X \leq \dim X'$.*
5. *Any two pseudo-closures of A have equal dimension.*
6. *If $A \subseteq X' \subseteq X$ then X' is a pseudo-closure of A.*

Suppose $\mathcal{M}'$ is an expansion of $\mathcal{M}$. Then $\mathcal{M}'$ is **approximable** over $\mathcal{M}$ (with respect to dim) if every $\mathcal{M}'$-definable set admits an $\mathcal{M}$-definable pseudo-closure.

The above definition admits an obvious generalization to theories. Let $L \subseteq L'$ be languages, T an L-theory equipped with an elementary rank, and T' an L'-theory with $T \subseteq T'$. We say that T' is **approximable over** T if $\mathcal{M}'$ is approximable over $\mathcal{M} = \mathcal{M}' \restriction L$ for all $\mathcal{M}' \models T'$.

We now return to the setting and notational conventions of Section 3. The following proposition is crucial:

Proposition 5.2. *Suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for all $i \in J$. If there is an $\mathcal{M}_\cap$-definable set X in which each X_i is pseudo-dense, then $(X_i)_{i \in J}$ is not separable. The converse implication holds provided $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$ for all $i \in J$.*

Proof. For the first statement, suppose X is a nonempty $\mathcal{M}_\cap$-definable subset of M^x in which each X_i is pseudo-dense, and $(X^i)_{i \in J}$ is a family of $\mathcal{M}_\cap$-definable sets satisfying $X_i \subseteq X^i$ for each $i \in J$. As X_i is pseudo-dense in X and disjoint from $X \setminus X^i$, we have $\dim X \setminus X^i < \dim X$ for all $i \in J$. Hence,
$$\dim \bigcup_{i \in J} (X \setminus X^i) < \dim X.$$
Thus $\dim \bigcap_{i \in J} X^i \geq \dim X$, so $\bigcap_{i \in J} X^i$ is nonempty.

Now suppose $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$ for each $i \in J$. Simplifying notation, we let $J = \{1, \dots, n\}$. Suppose X_i is an $\mathcal{M}_i$-definable set for each $1 \leq i \leq n$, and suppose there is no $\mathcal{M}_\cap$-definable set Z in which all of the X_i are pseudo-dense. We show $(X_i)_{i=1}^n$ is separable by applying simultaneous transfinite induction to $d_1, \ldots, d_n$ where d_i is the dimension of any pseudo-closure of X_i.

Let X^i be a pseudo-closure of X_i for each i and let

$$Z = X^1 \cap \ldots \cap X^n.$$

If $\dim X^j = -\infty$ for some $j \in J$, then X^j and Z are both empty, so $(X^i)_{i=1}^n$ separates $(X_i)_{i=1}^n$. If $\dim X^i = \dim Z$ for each i, then Lemma 5.1(2) shows each X_i is pseudo-dense in Z, contradiction. After re-arranging the X_i if necessary we suppose $\dim Z < \dim X^1$. Let $Y_1 = X_1 \cap Z$. As $(X_i)_{i=1}^n$ cannot be simultaniously pseudo-dense in an $\mathcal{M}_\cap$-definable set, it follows that $Y_1, X_2, \ldots, X_n$ cannot be simultaniously pseudo-dense in an $\mathcal{M}_\cap$-definable set. As the dimension of any pseudo-closure of Y_1 is strictly less then the dimension of X^1, an application of the inductive hypothesis provides $\mathcal{M}_\cap$-definable sets $Y^1, \ldots, Y^n$ separating $Y_1, X_2, \ldots, X_n$. It is easy to see

$$Y^1 \cup (X^1 \setminus Z), Y^2 \cap X^2, \ldots, Y^n \cap X^n$$

separates $X_1, \ldots, X_n$, which completes the proof. □

We say $\mathcal{M}_\cup$ is **approximately interpolative** if whenever $J \subseteq I$ is finite, $X_i \subseteq M^x$ is $\mathcal{M}_i$-definable for $i \in J$, and $(X_i)_{i\in J}$ are simultaneously pseudo-dense in some $\mathcal{M}_\cap$-definable set, then $\bigcap_{i\in J} X_i \neq \emptyset$. If the class of approximately interpolative models of $T_\cup$ is elementary, we call such an axiomatization the **approximate interpolative fusion** (of $(T_i)_{i\in I}$ over $T_\cap$).

The following corollary is an immediate consequence of Proposition 5.2.

Corollary 5.3. *If $\mathcal{M}_\cup$ is interpolative then it is approximately interpolative. The converse also holds if $\mathcal{M}_i$ is approximable over $\mathcal{M}$ for each $i \in I$.*

We say that T' **defines pseudo-denseness** over T if for every L-formula $\varphi(x, y)$ and every L'-formula $\varphi'(x, z)$, there is an L'-formula $\delta'(y, z)$ such that if $\mathcal{M}' \models T'$, $b \in M^y$, and $c \in M^z$, then

$$\varphi'(M^x, c) \text{ is pseudo-dense in } \varphi(M^x, b) \text{ if and only if } \mathcal{M}' \models \delta(b, c).$$

Theorem 5.4. *Suppose* dim *is an elementary rank on $T_\cap$. Then:*

1. *(1) if T_i defines pseudo-denseness over $T_\cap$ for all $i \in I$, then the approximate interpolative fusion exists;*
2. *(2) if in addition to conditions in (1), T_i is approximable over $T_\cap$, then the interpolative fusion exists.*

Proof. We first prove (1). Let $\varphi_\cap(x, y)$ be an $L_\cap$-formula, let $J \subseteq I$ be finite, and let $\varphi_i(x, z_i)$ be an L_i-formula for each $i \in J$. Let $\delta_i(y, z_i)$ be an L_i-formula defining pseudo-denseness for $\varphi_\cap(x, y)$ and $\varphi_i(x, z_i)$. For simplicity, we assume $J = \{1, \ldots, n\}$. Then we have the following axiom:

$$\forall y, z_1, \ldots, z_n \left(\left(\bigwedge_{i=1}^n \delta_i(y, z_i) \right) \to \exists x \bigwedge_{i=1}^n \varphi_i(x, z_i) \right).$$

Then $T_\cup$, together with one such axiom for each choice of $\varphi_\cap(x, y)$, J, and $\varphi_i(x, z_i)$ for $i \in J$ as above, axiomatizes the class of approximately interpolative $T_\cup$-models. Statement (2) follows from statement (1) and Corollary 5.3. □

We refer to the axiomatization given in the proof of Theorem 5.4 as the **pseudo-topological axioms**.

The following two issues deserve further investigation. First, can we say anything interesting about the model theory of approximate interpolative fusion without the conditions that T_i is approximable over $T_\cap$? Second, Theorem 5.4 tells us that defining pseudo-denseness is the key sufficient property for existence of the approximate interpolative fusion. We believe the converse may also be true, but we currently do not have a proof.

5.2. **Relativization to pseudo-cells.** In this subsection we consider variants of pseudo-denseness and approximate interpolative fusion.

Suppose $\mathcal{M}$ is an L-structure equipped with an ordinal rank dim. Throughout $\mathcal{C}$ is a collection of definable subsets of T-models. We say that an $\mathcal{M}$-definable set X admits a **$\mathcal{C}$-decomposition** if there is a finite family $(X_j)_{j \in J}$ from $\mathcal{C}$ such that

$$\dim\Big(X \triangle \bigcup_{j \in J} X^j\Big) < \dim X.$$

An $\mathcal{M}$-definable set X admits a **$\mathcal{C}$-patching** if there is a finite family $(X^j, Y^j, f^j)_{j \in J}$ such that for all $j, j' \in J$,

(1) Y^j is in $\mathcal{C}$.
(2) $f^j : X^j \to Y^j$ is an $\mathcal{M}$-definable bijection.
(3) And finally:
$$\dim\Big(X \triangle \bigcup_{j \in J} X^j\Big) < \dim X.$$

We say $\mathcal{C}$ is a **pseudo-cell collection** for $\mathcal{M}$ if either every $\mathcal{M}$-definable set admits a $\mathcal{C}$-decomposition or dim is preserved under $\mathcal{M}$-definable bijections and every $\mathcal{M}$-definable set admits a $\mathcal{C}$-patching. Examples of such $\mathcal{C}$ include the collection of almost irreducible sets in an $\aleph_0$-stable structure and the collection of irreducible varieties in an algebraically closed field.

The definition above naturally extends to theories. Let T be an L-theory equipped with an ordinal rank dim and $\mathcal{C}$ a collection of definable set in T-models. We say that $\mathcal{C}$ is a **pseudo-cell collection** for T if for all $\mathcal{M} \models T$, $\mathcal{C} \cap \mathrm{Def}(\mathcal{M})$ is a pseudo-cell collection for $\mathcal{M}$.

Suppose $L \subseteq L'$ are languages, T' an L'-theory with T the set of L-consequences, T is equipped with an ordinal rank dim, and $\mathcal{C}$ be a collection of definable sets in T-models. We say that T' **defines pseudo-denseness over** $\mathcal{C}$ if for every L'-formula $\varphi(x, y)$ and every L-formula $\varphi(x, z)$, there is an L'-formula $\delta'(y, z)$ such that if $\mathcal{M}' \models T'$, $c \in M^y$ with $\varphi(M^x, c) \in \mathcal{C}$ then

$$\varphi'(M^x, b) \text{ is pseudo-dense in } \varphi(M^x, c) \quad \text{if and only if} \quad \mathcal{M}' \models \delta'(b, c).$$

Clearly, if T' defines pseudo-denseness over T, then T' defines pseudo density over any collection $\mathcal{C}$ of definable sets of T-models. We are interested in the situation where the converse is true.

Let T be an L-theory and $\mathcal{C}$ be a collection of definable sets in T-models. We say that **defines $\mathcal{C}$-membership** if for every L-formula $\varphi(x, y)$ there is an L-formula $\gamma(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$,

$$\varphi(M^x, b) \text{ is in } \mathcal{C} \quad \text{if and only if} \quad \mathcal{M} \models \gamma(b).$$

We say T **defines dimension** if for all ordinal α, and L-formula $\varphi(x,y)$, there is a formula $\delta_\alpha(x,y)$ such that for all $\mathcal{M} \models T$and $b \in M^y$

$$\dim \varphi(M^x, b) = \alpha \quad \text{if and only if} \quad \mathcal{M} \models \delta_\alpha(b).$$

We leave the straightforward proof of the following proposition to the reader.

Proposition 5.5. *Suppose $\mathcal{C}$ is a collection of pseudo-cells, T defines $\mathcal{C}$-membership and dimension, and T' defines pseudo-denseness over $\mathcal{C}$. Then T' defines pseudo-denseness over T.*

Suppose dim is an ordinal rank on $\mathcal{M}_\cap$ and $\mathcal{C}$ is a collection of $\mathcal{M}_\cap$-definable sets. We say $\mathcal{M}_\cup$ is **$\mathcal{C}$-approximately interpolative** if for all finite $J \subseteq I$, $X_\cap \in \mathcal{C}$, and $(X_i)_{i \in J}$, where X_i is $\mathcal{M}_i$-definable and pseudo-dense in $X_\cap$, we have $\bigcap_{i\in J} X_i \neq \emptyset$. We call an axiomatization of the class of $\mathcal{C}$-approximately interpolative $T_\cup$-model, if it exists, the **$\mathcal{C}$-approximate interpolative fusion**. Clearly, if $\mathcal{M}_\cup$ is approximately interpolative then it is $\mathcal{C}$-approximately interpolative. The following Proposition gives situations where the converse is true. We omit the straightforward proof.

Proposition 5.6. *Suppose $\mathcal{C}$ is a collection of pseudo-cells in $\mathcal{M}_\cap$. Then we have the following:*

1. *$\mathcal{M}_\cup$ is approximately interpolative if and only if it is $\mathcal{C}$-approximately interpolative;*
2. *if moreover $\mathcal{M}_i$ is approximable over $\mathcal{M}_\cap$, then $\mathcal{M}_\cup$ is interpolative if and only if it is $\mathcal{C}$-approximately interpolative.*

We next turn to the axiomatization problem:

Theorem 5.7. *Suppose* dim *is an ordinal rank on $T_\cap$, $\mathcal{C}$ is a collection of definable sets of $T_\cap$-models such that $T_\cap$ defines $\mathcal{C}$-membership, and T_i defines pseudo-denseness over $\mathcal{C}$ for $i \in I$. Then we have the following:*

1. *The $\mathcal{C}$-approximate interpolative fusion exists;*
2. *if $\mathcal{C}$ is a pseudo-cell collection for $T_\cap$, then the approximate interpolative fusion exists;*
3. *if in addition, T_i is approximable over $T_\cap$ for each $i \in I$, then the interpolative fusion exists.*

Proof. We first prove statement (1). Let $\varphi_\cap(x,y)$ be an $L_\cap$-formula, let $J \subseteq I$ be finite, and let $\varphi_i(x,z_i)$ be an L_i-formula for each $i \in J$. Let $\gamma_\cap(y)$ be an $L_\cap$-formula defining $\mathcal{C}$-membership for $\varphi_\cap(x,y)$ and $\delta_i(y,z_i)$ an L_i-formula defining pseudo-denseness over $\mathcal{C}$ for $\varphi_\cap(x,y)$ and $\varphi_i(x,z_i)$ for each $i \in J$. For simplicity, we assume $J = \{1,\dots,n\}$. Then we have the following axiom:

$$\forall y, z_1, \dots, z_n \left(\left(\gamma_\cap(y) \wedge \bigwedge_{i=1}^{n} \delta_i(y, z_i) \right) \to \exists x \bigwedge_{i=1}^{n} \varphi_i(x, z_i) \right).$$

Then $T_\cup$, together with one axiom of the above form for each choice of $\varphi_\cap(x,y)$, J, and $\varphi_i(x,z_i)$ for $i \in J$ as above, axiomatizes the class of $\mathcal{C}$-approximately interpolative $T_\cup$-models. Assertions (2) and (3) follows immediately from Corollary 5.6. ☐

The axiomatization given in the proof of Theorem 5.7 is slightly different than that of Theorem 5.4. They are nevertheless very similar in spirit, so we also refer to the former as the **pseudo-topological axioms**.

5.3. **Tame topological base.** If $\mathcal{M}$ is o-minimal, then $\mathcal{M}'$ is approximable over $\mathcal{M}$ if and only if every $\mathcal{M}'$-definable closed set is $\mathcal{M}$-definable. This equivalence only depends on two well-known facts from o-minimality. One of these is known as the frontier inequality, we refer to the other as the residue inequality. We explore these issues in an abstract setting below.

A **definable topology** $\mathcal{T}$ on $\mathcal{M}$ consists of a Hausdorff topology $\mathcal{T}_x$ on each M^x for which there is an L-formula $\varphi(x,y)$ such that $\{\varphi(M^x,a) : a \in M^y\}$ is an open basis for $\mathcal{T}_x$. For the rest of the section, we suppose $\mathcal{T}$ is a definable topology and dim is an elementary rank on $\mathcal{M}$. We also suppose that dim is definable in families. Note that we obtain a definable topology satisfying the same properties on any T-model.

Let A be a subset of M^x. We denote by $\mathrm{cl}(A)$ is the closure of A with respect to $\mathcal{T}_x$. The **frontier** $\mathrm{fr}A$ of A is defined as $\mathrm{cl}(A) \setminus A$. We say that A **has interior** in $X \subseteq M^x$ if there is an open $U \subseteq M^x$ such that $U \cap X \subseteq A$. It is easy to see that the closure, interior, and frontier of a definable subset of M^x are definable.

In general there need be no connection between pseudo-denseness and $\mathcal{T}$-denseness. We give conditions under which the two naturally relate. We say $\mathcal{M}$ satisfies the **frontier inequality** (with respect to $\mathcal{T}$ and dim) if

$$\dim \mathrm{fr}(X) < \dim X \quad \text{for all definable } X.$$

This is a strong assumption which in particular implies, via a straight-forward induction on dimension, that every definable set is a Boolean combination of open definable sets.

Lemma 5.8. *Suppose $\mathcal{M}$ satisfies the frontier inequality and $X' \subseteq X$ are $\mathcal{M}$-definable subsets of M^x. If $\dim X' = \dim X$ then X' has interior in X.*

Proof. Suppose $\dim X' = \dim X$. If X' has empty interior in X, then $X \setminus X'$ is dense in X, and so X is contained in the closure of $X \setminus X'$. In particular X' is contained in the closure of $X \setminus X'$. The frontier inequality implies $\dim X' < \dim X \setminus X' \leq \dim X$, contradiction. □

Lemma 5.9. *The following are equivalent:*

1. *$\mathcal{M}$ satisfies the frontier inequality,*
2. *if $A \subseteq M^x$ is dense in a definable $X \subseteq M^x$ then A is pseudo-dense in X.*

Proof. Suppose that $\mathcal{M}$ does not satisfy the frontier inequality. Let $X \subseteq M^x$ be definable and suppose $\dim \mathrm{fr}(X) \geq \dim X$. It follows that X is dense in $\mathrm{cl}(X)$ and not pseudo-dense in $\mathrm{cl}(X)$.

Now suppose that $\mathcal{M}$ satisfies the frontier inequality and that $A \subseteq M^x$ is dense in a definable set $X \subseteq M^x$. Suppose $X' \subseteq X$ is definable and $\dim X' = \dim X$. Lemma 5.8 implies that X' has nonempty interior in X. Thus A intersects X'. It follows that A is pseudo-dense in X. □

The converse to (2) above almost always fails for general definable sets X. If $X' \subseteq M^x$ is an infinite definable set and $p \in M^x$ does not lie in $\mathrm{cl}(X')$ then X' is pseudo-dense in but not dense in $X = X' \cup \{p\}$.

We say that a definable $X \subseteq M^x$ is **dimensionally pure** (with respect to $\mathcal{T}$ and dim) if one of the following equivalent conditions holds:

(1) there are no nonempty $\mathcal{M}$-definable sets X^1 and X^2 such that $X = X^1 \cup X^2$, X^1 and X^2 are closed in X, neither X^1 nor X^2 contains the other, and $\dim X^1 \neq \dim X^2$,
(2) if $U \subseteq X$ is definable, nonempty, and open in X then $\dim U = \dim X$.

We only use (2), so we leave it to the reader to show that the two definitions above are equivalent.

Lemma 5.10. *Suppose $X \subseteq M^x$ is definable. Then the following are equivalent:*

(1) X is dimensionally pure,
(2) if a subset A of M^x is pseudo-dense in X then A is dense in X.

Proof. Suppose X is not dimensionally pure. Let U be a definable nonempty open subset of X such that $\dim U < \dim X$. Then $X \setminus U$ is pseudo-dense in X and not dense in X.

Suppose X is dimensionally pure and A is pseudo-dense in X. Suppose U is a nonempty open subset of X. Then there is a definable nonempty open subset U' of U. Then $\dim U' = \dim X$ so A intersects U'. Hence A is dense in X. □

A family $(X^i)_{i=1}^n$ of $\mathcal{M}$-definable sets is a **dimensionally pure decomposition** of X if each X^i is dimensionally pure, $X = \bigcup_{i=1}^n X^i$, and

$$\dim X^1 < \dim X^2 < \ldots < \dim X^n.$$

The following proposition justifies the use of this term.

Proposition 5.11. *The following are equivalent:*

(1) for every definable $X \subseteq M^x$ there is a definable $Y \subseteq X$ such that $\dim Y < \dim X$ and $X \setminus Y$ is dimensionally pure.
(2) every definable $X \subseteq M^x$ is a finite union of dimensionally pure sets,
(3) every definable $X \subseteq M^x$ admits a dimensionally pure decomposition.

The proof of Proposition 5.11 requires Lemma 5.12 below.

Lemma 5.12. *Suppose $X_1, \ldots, X_n$ are dimensionally pure and*

$$\dim X_1 = \ldots = \dim X_n.$$

Then $X = \bigcup_{k=1}^n X_k$ is dimensionally pure.

Proof. Note $\dim X = \dim X_k$ for all $1 \leq k \leq n$. Let U be a definable, nonempty, open subset of X. Fix $p \in U$. Suppose $p \in X_k$ for fixed $1 \leq k \leq n$. As X_k is dimensionally pure we have $\dim(U \cap X_k) = \dim X_k = \dim X$, so $\dim U = \dim X$. □

We now prove Proposition 5.11.

Proof. Straightforward induction on dimension shows that (1) implies (3) and (3) trivially implies (2). It follows directly from the definition of a dimensionally pure decomposition that (3) implies (1).

We show that (2) implies (1). Suppose that X is a union of dimensionally pure definable sets $Y^1, \ldots, Y^m$. Apply induction on $\dim X$. Suppose without loss of generality $1 \leq j \leq m$ is such that $\dim Y_k = \dim X$ when $k \leq j$ and $\dim Y_k < \dim X$ when $k > j$. Lemma 5.12 implies $X^n := \bigcup_{k=1}^j Y_k$ is dimensionally pure. As $\dim X \setminus X^n < \dim X$ induction implies $X \setminus X^n$ has a dimensionally pure decomposition $X^1, \ldots, X^{n-1}$. Then $X^1, \ldots, X^n$ is a dimensionally pure decomposition of X. □

Let $X \subseteq M^x$ be definable. Given $p \in X$ we declare

$$\dim_p X = \min\{U \cap X : U \text{ a nonempty definable neighbourhood of } p\}.$$

Note that X is dimensionally pure if and only if $\dim_p X = \dim X$ for all $p \in X$. We declare $X[k]$ to be $\{p \in X : \dim_p X = k\}$. We say that

$$\operatorname{rs}(X) = \{p \in X : \dim_p X < \dim X\}$$

is the **residue** of X. As $\mathcal{T}_x$ admits a definable basis, and dim is definable in families, it follows that $\operatorname{rs}(X)$ and each $X[k]$ is definable.

Lemma 5.13. *Suppose that $\mathcal{M}$ satisfies the frontier inequality. Then the following are equivalent for every definable $X \subseteq M^x$:*

(1) $\dim \operatorname{rs}(X) < \dim X$.
(2) there is a definable $Y \subseteq X$ such that $\dim Y < \dim X$ and $X \setminus Y$ is dimensionally pure.

Proof. Suppose (1) holds. Let Y be the closure of $\operatorname{rs}(X)$ inside X. Then $X \setminus Y$, the interior of $X[\dim X]$ inside X, is dimensionally pure. The frontier inequality, together with (1) implies $\dim Y < \dim X$.

Suppose (1) fails. We let $Y \subseteq X$ be definable such that $\dim Y < \dim X$ and show $X \setminus Y$ is not dimensionally pure. The frontier inequality implies that

$$\dim \operatorname{cl}(Y) \cap X < \dim X.$$

As $\dim \operatorname{rs}(X) = \dim X$ there is a $p \in X \setminus \operatorname{cl}(Y)$ such that $\dim_p X < \dim X$. It follows that there is a definable open subset U of X such that $p \in U$, $U \cap Y = \emptyset$, and $\dim U < \dim X$. Thus $X \setminus Y$ is not dimensionally pure. □

We say that $\mathcal{M}$ satisfies the **residue inequality** (with respect to $\mathcal{T}$ and dim) if

$$\dim \operatorname{rs}(X) < \dim X \quad \text{for all definable } X.$$

Note that the residue inequality implies that all definable discrete sets are finite..

We say $\mathcal{T}$ is dim**-compatible** if $\mathcal{M}$ satisfies both the frontier inequality and the residue inequality. Lemma 5.13 and Proposition 5.11 show that $\mathcal{T}$ is dim-compatible if and only if $\mathcal{M}$ satisfies the frontier inequality and every definable set admits a dimensionally pure decomposition. **For the remainder of Section 5.3** dim **is an elementary rank on $\mathcal{M}$ and $\mathcal{T}$ is a** dim**-compatible definable topology on $\mathcal{M}$.** Definibility of dimension, and the existence of a definable basis ensure that dim-compatbility is a first order property, i.e. the topology on any model of the theory of $\mathcal{M}$ is dim-compatible.

Proposition 5.14. *Suppose $X \subseteq M^x$ is $\mathcal{M}$-definable and $A \subseteq M^x$. Then A is pseudo-dense in X if and only if A is dense in the maximal component of a dimensionally pure decomposition of X.*

Proof. Suppose $(X^i)_{i=1}^n$ is a dimensionally pure decomposition of X. Suppose A is dense in X^n. Then Lemma 5.9 implies A is pseudo-dense in X^n. As $\dim X \setminus X^n < \dim X$ it follows that A is pseudo-dense in X^n. Now suppose A is pseudo-dense in X. As $\dim X \setminus X^n < \dim X$ it follows that A is pseudo-dense in X^n. An application of Lemma 5.10 shows A is dense in X^n. □

Proposition 5.15. *Let $(X_b)_{b \in M^y}$ be a definable family of subsets of M^x. Then there are definable families $(X_b^1)_{b \in M^y}, \ldots, (X_b^n)_{b \in M^y}$ such that $X_b^1, \ldots, X_b^n$ is a dimensionally pure decomposition of X_b for all b.*

Proof. After applying definability of dimension in families and partitioning $(X_b)_{b\in M^y}$ into finitely many definable subfamilies if necessary we suppose $\dim X_b$ is constant. We apply induction to $\dim X_b$. If $\dim X_b \leq 0$ then X_b is finite, hence dimensionally pure. Suppose $\dim X_b > 0$. Let Y_b be the closure of $X_b[<]$ inside X_b and $X^n_b = X_b \setminus Y_b$. It follows from the existence of a definable basis and the definability of dim that $(X^n_b)_{b\in M^y}$ is a definable family. Then X^n_b is dimensionally pure and $\dim Y_b < \dim X_b$. Now apply induction to obtain a uniform dimensionally pure decomposition $(X^1_b, \dots, X^{n-1}_b)_{b\in M^y}$ of $(Y_b)_{b\in M^y}$. □

Theorem 5.16. *Any expansion T' of T defines pseudo-denseness over T.*

Proof. Suppose $\mathcal{M}$ is a T-model and $\mathcal{M}'$ is a T'-model expanding $\mathcal{M}$. Suppose $(X_a)_{a\in M^y}$ and $(X'_b)_{b\in M^z}$ are $\mathcal{M}, \mathcal{M}'$-definable families of subsets of M^z, respectively. It follows from Corollary 5.14 that X'_b is pseudo-dense in X_a if and only if X'_b is dense in the maximal component of a dimensionally pure decomposition of X_a. Proposition 5.15 yields definable families $(X^1_a)_{a\in M^y}, \dots, (X^n_a)_{a\in M^y}$ such that $X^1_a, \dots, X^n_a$ is a dimensionally pure decomposition of X_a for all $a \in M^y$. Thus X'_b is pseudo-dense in X_a if and only if X'_b is dense in X^n_a. The existence of a definable basis for $\mathcal{T}_x$ shows the set of (a, b) such that X'_b is dense in X^n_a is definable. □

Theorem 5.17. *Suppose $\mathcal{M}'$ expands $\mathcal{M}$. Then $\mathcal{M}'$ is approximable over $\mathcal{M}$ if and only if the closure of any $\mathcal{M}'$-definable set is $\mathcal{M}$-definable.*

Proof. Let $\mathrm{cl}(A)$ be the $\mathcal{T}_x$-closure of $A \subseteq M^x$. Lemma 5.9 gives the backward direction. For the forward direction, suppose $\mathcal{M}'$ is approximable over $\mathcal{M}$ and $X' \subseteq M^x$ is $\mathcal{M}'$-definable. We apply induction to the dimension of a pseudo-closure of X'. Let X be a pseudo-closure of X' and $(X^i)^n_{i=1}$ be a dimensionally pure decomposition of X. If $\dim X = -\infty$ then X' is empty and trivially $\mathcal{M}$-definable. Suppose $\dim X \geq 0$. We have

$$\mathrm{cl}(X') = \bigcup_{i=1}^{n} \mathrm{cl}(X' \cap X^i).$$

As $X' \cap X^i \subseteq X^i$, any pseudo-closure of $X' \cap X^i$ has dimension $\leq \dim X^i$. If $i \leq n-1$ then $\mathrm{cl}(X' \cap X^i)$ is $\mathcal{M}$-definable by induction. On the other hand, X' is dense in X^n by Corollary 5.14, and so $\mathrm{cl}(X' \cap X^n) = \mathrm{cl}(X^n)$ is $\mathcal{M}$-definable. Thus $\mathrm{cl}(X')$ is a finite union of $\mathcal{M}$-definable sets, hence $\mathcal{M}$-definable. □

We conclude this section by giving examples of structures with compatible definable topologies. In each case $\mathcal{T}$ and dim are canonical, so we do not describe them in detail. And in each case the existence of dimensionally pure decompositions follows from the appropriate cell decomposition or "weak cell decomposition" result. In different settings cells (or "weak cells") have different definitions, but they are easily seen to be dimensionally pure in each case.

The most familiar case is when $\mathcal{M}$ is an o-minimal expansion of a dense linear order, see [vdD98]. More generally, it follows from [SW15, Proposition 4.1,4.3] that if $\mathcal{M}$ is a dp-minimal expansion of a divisible ordered abelian group then the usual order topology is compatible. This covers the case when $\mathcal{M}$ is an expansion of an ordered abelian group with weakly o-minimal theory. It is shown in Johnson's thesis [Joh16] that a dp-minimal, non strongly minimal, expansion of a field admits a definable field topology and it is shown in [SW15] that this topology is compatible.

It follows in particular that a C-minimal expansion of an algebraically closed field, or a P-minimal expansion of a p-adically closed field admits a compatible definable topology. It was previously shown in [CKDL17] that P-minimal expansions of p-adically closed fields satisfy the frontier inequality and admit dimensionally pure decompositions.

We say that T is an **open core** of T' if the closure of every T'-definable set in every T'-model $\mathcal{M}'$ is $\mathcal{M} = \mathcal{M}'|L$ definable. Theorem 5.16 and Theorem 5.17 together yield the following theorem.

Theorem 5.18. *If $T_\cap$ admits an elementary rank* dim *and a* dim*-compatible definable topology, and $T_\cap$ is an open core of T_i for each $i \in I$, then $T^*_\cup$ exists. In particular, if $T_\cap$ is an o-minimal expansion of a dense linear order or a p-minimal expansion of a p-adically closed field, and $T_\cap$ is an open core of T_i for each $i \in I$, then $T^*_\cup$ exists.*

5.4. **$\aleph_0$-stable base.** We assume throughout this subsection that T is $\aleph_0$-stable, dim is the Morley rank on T, and mult is Morley degree on T.

Suppose X^1 and X^2 are $\mathcal{M}$-definable subsets of M^x. Then X^1 is **almost a subset** of X^2 if

$$\dim(X^1 \setminus X^2) < \dim(X^1),$$

and X^1 is **almost equal** to X^2 if X^1 is almost a subset of X^2 and vice versa. A subset X of M^x is **almost irreducible** if X is $\mathcal{M}$-definable and whenever $X = X^1 \cup X^2$ with $\mathcal{M}$-definable X_1 and X_2 we must have

$$\text{either} \quad \dim(X \setminus X^1) < \dim X \quad \text{or} \quad \dim(X \setminus X_2) < \dim X.$$

An $\mathcal{M}$-definable set of Morley degree one is almost irreducible. The converse is not true in general but holds when $\mathrm{Th}(\mathcal{M})$ defines Morley rank or when $\mathcal{M}$ is $\aleph_0$-saturated. The main advantages of the assumption that T is $\aleph_0$-stable is reflected by the following easy proposition:

Lemma 5.19. *Suppose A is a subset of M^x. Then an $\mathcal{M}$-definable set $X \subseteq M^x$ is the pseudo-closure of A if and only if $A \subseteq X$ and*

$$(\dim X, \mathrm{mult} X) \le_{\mathrm{Lex}} (\dim X', \mathrm{mult} X')$$

for all $\mathcal{M}$-definable $X' \subseteq M^x$ with $A \subseteq X'$.

Proof. By definition, A is pseudo-dense in X if and only if for all $\mathcal{M}$-definable $Y \subseteq X$ such that $A \cap Y = \varnothing$ implies $\dim Y < \dim X$. As $\mathcal{M}$ is $\aleph_0$-stable, the latter is also equivalent to the statement that all $\mathcal{M}$-definable set $X' \subseteq X$ such that $A \subseteq X'$ must have $\dim X' = \dim X$ and $\mathrm{mult} X' = \mathrm{mult} X$. The desired conclusion follows from Lemma 5.1. □

The preceding lemma has the following important immediate consequence for the approximability condition in this setting.

Proposition 5.20. *Every $A \subseteq M^x$ has a pseudo-closure. Hence every expansion of $\mathcal{M}$ is approximable over $\mathcal{M}$ and every expansion of T is approximable over T.*

Proof. The desired conclusions are immediate consequences of Lemma 5.1 and Lemma 5.19. □

Corollary 5.21. *If $\mathcal{M}_\cap$ is $\aleph_0$-stable and* dim *is Morley rank on $\mathcal{M}_\cap$, then $\mathcal{M}_\cup$ is an interpolative fusion if and only it is an approximately interpolative fusion.*

The reader might wonder if we get another free-ride elsewhere. In the most interesting situations, this question unfortunately has a negative answer. If $\dim_1, \dim_2$ are elementary ranks on an $L^\diamond$-theory $T^\diamond$ then we say $\dim_1$ is **smaller then** $\dim_2$ if $\dim_1 A \leq \dim_2 A$ always holds.

Lemma 5.22. *The theory $T^\diamond$ is $\aleph_0$-stable if and only if it admits an elementary rank* $\dim$ *such that for every $T^\diamond$-model $\mathcal{M}^\diamond$, $\mathcal{M}^\diamond$-definable set X, and family $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint $\mathcal{M}^\diamond$-definable subsets of X, we have $\dim X_n < \dim X$ for some n. If $T^\diamond$ is $\aleph_0$-stable then Morley rank is the smallest elementary rank with this property.*

Proof. It is well-known that Morley rank RM is an ordinal rank satisfying the necessary conditions when $T^\diamond$ is $\aleph_0$-stable. Suppose dim is an elementary rank satisfying the required properties. We will show that $\mathrm{RM}(X) \leq \dim X$ for all $\mathcal{M}^\diamond$-definable sets X in $T^\diamond$-models $\mathcal{M}^\diamond$. This implies that RM is ordinal valued and hence that $T^\diamond$ is $\aleph_0$-stable.

As RM and dim are elementary it suffices to fix an $\aleph_0$-saturated $T^\diamond$-model $\mathcal{M}^\diamond$ and show $\mathrm{RM}(X) \leq \dim(X)$ for all $\mathcal{M}^\diamond$-definable sets X. We apply transfinite induction to $\mathrm{RM}(X)$. If $\mathrm{RM}(X) = -\infty$ then X is empty and $\dim X = -\infty$. If $\mathrm{RM}(X) = 0$ then X is finite and $\dim X = 0$. We now treat the inductive case. We fix an ordinal $\alpha < \mathrm{RM}(X)$ and show $\alpha < \dim X$. As $\mathcal{M}^\diamond$ is $\aleph_0$-saturated there are pairwise disjoint $\mathcal{M}$-definable subsets $(X_n)_{n\in\mathbb{N}}$ of X such that $\mathrm{RM}(X_n) \geq \alpha$ for all n. Induction implies $\mathrm{RM}(X_n) \leq \dim X_n$ hence $\dim X_n \geq \alpha$ for all n. It follows by our assumption on dim that $\dim X > \alpha$. □

Proposition 5.23. *Suppose $L^\diamond$ is countable and* $\dim^\diamond$ *is an elementary rank on a complete $L^\diamond$-theory $T^\diamond$. If $T^\diamond$ is not $\aleph_0$-stable then there is an expansion of $T^\diamond$ which is not approximable over $T^\diamond$.*

Proof. Suppose $T^\diamond$ is not $\aleph_0$-stable. Applying Lemma 5.22, we obtain a $T^\diamond$-model $\mathcal{N}^\diamond$, an $\mathcal{N}^\diamond$-definable set X with $\dim^\diamond X = \alpha$, and a sequence $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint $\mathcal{N}^\diamond$-definable subsets of X such that $\dim^\diamond X_n = \alpha$ for all n. Let $\mathcal{M}^\diamond$ be a countable elementary submodel of $\mathcal{N}^\diamond$ such that X and every X_n is $\mathcal{M}^\diamond$-definable. After replacing $\mathcal{N}^\diamond$ with $\mathcal{M}^\diamond$ if necessary we suppose $\mathcal{N}^\diamond$ is countable.

Given $S \subseteq \mathbb{N}$ let $A_S = \bigcup_{n\in S} X_n$. We show that A_S does not have a pseudo-closure for uncountably many $S \subseteq \mathbb{N}$. Suppose $S \subseteq \mathbb{N}$ is nonempty and X' is a pseudo-closure of A_S. As $A_S \subseteq X$ we have $\dim^\diamond X' \leq \dim^\diamond X$. As S is nonempty we have $X_n \subseteq X'$ for some n, so $\dim^\diamond X' \geq \dim^\diamond X$. Thus any pseudo-closure X' of A_S has $\dim^\diamond X' = \alpha$.

Suppose $S, S' \subseteq \mathbb{N}$ are nonempty and $S \subsetneq S'$. We show any pseudo-closure of X_S is not a pseudo-closure of $A_{S'}$. Fix $n \in S \setminus S'$. Suppose X' is a pseudo-closure of A_S. Then $\dim^\diamond X' = \alpha$, X_n is an $\mathcal{N}^\diamond$-definable subset of X' with $\dim^\diamond X_n = \alpha$, but X_n is disjoint from $A_{S'}$. Thus X' is not a pseudo-closure of $A_{S'}$.

Let $\mathfrak{J}$ be an uncountable collection of nonempty subsets of $\mathbb{N}$ such that $S \subsetneq S'$ for all distinct $S, S' \in \mathfrak{J}$. If $S, S' \in \mathfrak{J}$ are distinct then A_S and $A_{S'}$ cannot have a common pseudo-closure. As $\mathcal{N}^\diamond$ and L are countable there are only countably many $\mathcal{N}^\diamond$-definable sets so there are uncountably many $S \in \mathfrak{J}$ such that A_S does not have a pseudo-closure. The expansion of $\mathcal{N}^\diamond$ by a predicate defining any such A_S is not approximable over $\mathcal{N}^\diamond$. From here, we can construct in an obvious way an expansion of $T^\diamond$ which is not approximable over $T^\diamond$. □

See Section 6.6 for a concrete non-approximable expansion of the theory of $(\mathbb{Z};+)$.

We next turn to the problem of understanding definability of pseudo-denseness in this setting. Lemma 5.19 motivates the following definition. Suppose M' is a model of T', $\mathcal{M} = \mathcal{M}' \restriction L$, and $X' \subseteq M^x$. Define

$$\dim' X' = \dim X \quad \text{and} \quad \mathrm{mult}' X' = \mathrm{mult} X$$

where X is a pseudo-closure of X'. This gives us an ordinal rank on T' which we will refer to as the **induced rank** on T'. In general, the induced rank on T' might not even be elementary. The following Corollary is an immediate consequence of Lemma 5.19.

Lemma 5.24. *For $A \subseteq M^x$ and $\mathcal{M}$-definable $X \subseteq M^x$, we have the following:*

(1) *A is pseudo-dense in X if and only if we have both* $\dim'(X \cap A) = \dim(X)$ *and* $\mathrm{mult}'(X \cap A) = \mathrm{mult}(X)$*;*
(2) *A is pseudo-dense in an almost irreducible set X if and only if* $\dim'(X \cap A)$ *is the same as* $\dim(X)$.

We say T **defines multiplicity** (or has the **DMP**) if for all L-formulas $\varphi(x,y)$, ordinals α, and n, there is an L-formula $\mu_{\alpha,n}(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$ we have that

$$\mathcal{M} \models \mu_{\alpha,n}(b) \text{ if and only if } \dim \varphi(M^x, b) = \alpha \text{ and } \mathrm{mult}\ \varphi(M^x, b) = n.$$

In particular, if T defines multiplicity then T defines Morley rank. Let T' be an expansion of T.

Proposition 5.25. *Suppose T defines multiplicity. Then T' defines pseudo-denseness over T if and only if T' defines induced dimension.*

Proof. Suppose T' defines pseudo density and $\varphi'(x,y)$ is an L'-formula. Let $(X'_{b'})_{b' \in Y'}$ be the family of subsets of M^x defined by $\varphi'(x,y)$. Using the assumption that T' defines pseudo density and a usual compactness argument, we obtain a family $(X_c)_{c \in Z}$ defined by a formula who choice might depend on $\varphi(x,y)$ but not on $\mathcal{M}'$ such that for every $b' \in Y$, X'_b is pseudo-dense in a member of the family $(X_c)_{c \in Z}$. It follows from Lemma 5.20 that $\dim'(X'_{b'}) = \alpha$ for $b' \in Y$ if and only there is $c \in Z$ such that X'_b is pseudo-dense in X_c and $\dim(X_c) = \alpha$. As T defines multiplicity and T' defines pseudo-denseness, it follows that T' defines induced dimension.

Now suppose T' defines induced dimension. Let $\mathcal{C}$ be the collection of sets of almost irreducible subsets of T-models. Then $\mathcal{C}$ is a collection of pseudo-cells. As T defines multiplicity, T defines $\mathcal{C}$-membership. So by Proposition 5.5, it suffices to show T' defines pseudo-denseness over $\mathcal{C}$. Let $(X'_{b'})_{b' \in Y'}$ and $(X_c)_{c \in Z}$ be a families defined by an L'-formula $\varphi'(x,y)$ and an L-formul $\varphi(x,y)$. It follows from Lemma 5.19 that when X_c is in $\mathcal{C}$, $X'_{b'}$ is pseudo-dense in X_c if and only if $\dim'(X \cap X') = \dim(X)$. The desired conclusion follows. □

Remark 5.26. Proposition 5.25 is mainly of interest because there are natural situations where the induced elementary rank on T' is a natural notion of dimension in that setting. One example is when T is ACF and T' is ACVF (see Section 6.5). Proposition 5.31 below generalizes this example.

Remark 5.27. There is also an analogue of Proposition 5.25 which involves both $\dim'$ and mult' and without assumming that T defines multiplicity, we do not include it here as there is no natural example.

Theorem 5.28. *Suppose $T_\cap$ is $\aleph_0$-stable and defines multiplicity. If T_i defines induced dimension over $T_\cap$ then $T^*_\cup$ exists.*

Proof. This is an immediate consequence of Theorem 5.4, Proposition 5.20, and Proposition 5.25. □

5.4.1. *Algebraic Dimension.* Let $\mathcal{N}$ be an arbitrary one-sorted structure and $X \subseteq N^m$ be definable. The **algebraic dimension** $\operatorname{adim}(X)$ of X is the maximal k for which there is $a = (a_1, \dots, a_n) \in X(\boldsymbol{\mathcal{N}})$ such that (after permuting coordinates) $a_1, \dots, a_k$ are acl-independent over N. It is well known that algebraic dimension gives an elementary rank on models of the theory of $\mathcal{N}$. The following fact is also well known.

Fact 5.29. A theory defines algebraic dimension if and only if it eliminates $\exists^\infty$.

Lemma 5.30. *Suppose T is strongly minimal and* acl' *agrees with* acl *over all T'-models. Then T' defines induced dimension if and only if T' eliminates $\exists^\infty$.*

Proof. $\mathcal{M}' \models T'$, $\mathcal{M} = \mathcal{M}' \restriction L$ and $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}' \restriction L$, X' an arbitrarily $\mathcal{M}$-definable subset of M^x, $\dim'$ is the induced dimension on T', adim and adim' are the algebraic dimension in $\mathcal{M}$ and in $\mathcal{M}'$. Using Fact 5.29, it suffices to show that $\dim' = \operatorname{adim}'$. As T is strongly minimal, dim coincides with adim. Hence,

$$\dim'(X') = \min\{\operatorname{adim}(X) \mid X \subseteq M^x \text{ is } \mathcal{M}\text{-definable, and } X' \subseteq X\}.$$

As $\operatorname{acl}' = \operatorname{acl}$, whenever $a \in X'(\boldsymbol{\mathcal{M}}')$ has k components which are acl'-independent over M, these components are also acl-independent over M as well. Hence, $\operatorname{adim}'(X') \leq \dim'(X')$. Let $X \subseteq M^x$ be a pseudo-closure of X', and $n = \operatorname{adim} X$. Suppose $a' \in X'(\boldsymbol{\mathcal{M}}')$ has less than n components which are acl'-indenpendent. As $\operatorname{acl}' = \operatorname{acl}$ and algebraic dependence can be expressed in first-order logic, there is $\mathcal{M}$-definable $Y \subseteq M^x$ with $a' \in Y(\boldsymbol{\mathcal{M}})$ and

$$\dim Y = \operatorname{adim} Y < n = \operatorname{adim} X = \dim X$$

Using the fact that X is the pseudo-closure of X' and compactness, we get $a \in X'(\boldsymbol{\mathcal{M}}')$ such that $a \notin Y(\boldsymbol{\mathcal{M}})$ for any $Y \subseteq M^x$ with $\dim Y < \dim X$. Thus $\operatorname{adim}'(X') \geq \dim'(X')$ and the desired conclusion follows. □

Proposition 5.31. *Let $T_\cap$ be model complete, strongly minimal, and define multiplicity. If for all $i \in I$, T_i eliminates $\exists^\infty$ and* acl_i *agrees with* $\operatorname{acl}_\cap$*, then $T^*_\cup$ exists.*

Proof. This follows immediately from Theorem 5.28 and Lemma 5.30. □

5.5. **Toward $\aleph_0$-categorical base.** Througout this subsection, we assume that α, β, γ are ordinals, L has finitely many sorts, T is $\aleph_0$-stable and $\aleph_0$-categorical, dim is Morley rank on T, and mult is Morley degree on T. We make extensive use of Proposition 5.23, which ensures that every subset of M^x is approximable. Despite this, we consider this subsection more of a first step toward developing the theory of interpolative fusions over an $\aleph_0$-categorical base rather than the continuation of the preceding section. A full-fledged theory should also cover Proposition 4.16.

In a distinct line of idea from Theorem 5.28, the $\aleph_0$-stable assumption also gives us the following "inductive" procedure to check whether a subset is pseudo-dense in an almost irreducible set:

Lemma 5.32. *Suppose $X \subseteq M^x$ is almost irreducible, $\mathcal{D}$ is a collection of almost irreducible subsets of M^x such that any almost irreducible subset of M^x is almost equal to an element in $\mathcal{D}$, and A is a subset of M^x. For $\alpha < \dim X$, let $\mathcal{D}_\alpha(A, X)$ be the collection of almost irreducible $X_\alpha \in \mathcal{D}$ such that*

$$\dim X_\alpha = \alpha, \ A \text{ is pseudo-dense in } X_\alpha, \text{ and } X_\alpha \text{ is almost a subset of } X.$$

If $\mathcal{D}_\beta(A, X) = \emptyset$ for all $\alpha < \beta < \dim X$, then we have the following:

1. *If $\mathcal{D}_\alpha(A, X)$ is infinite up to almost equality, then A is pseudo-dense in X.*
2. *If $X^1_\alpha, \dots, X^n_\alpha$ are the representatives of the equivalence classes of $\mathcal{D}_\alpha(A, X)$ by almost equality, and*
$$A' := A \setminus \bigcup_{i=1}^{n} X^i_\alpha \text{ is pseudo-dense in } X,$$
then $\mathcal{D}_\beta(A', X) = \emptyset$ for all $\alpha \le \beta < \dim X$, and A is pseudo-dense in X if and only if so is A'.

Proof. As $\mathcal{M}$ is $\aleph_0$-stable, $A \cap X$ has a pseudo-closure Y which is a subset of X by Lemma 5.20. Suppose $\mathcal{D}_\beta(A, X) = \emptyset$ for all $\alpha < \beta < \dim X$. Then either $\dim Y \le \alpha$ or $\dim Y = \dim X$. If $\mathcal{D}_\alpha(A, X)$ is infinite up to almost equality, then $\dim Y > \alpha$, and so $\dim Y = \dim X$. The latter implies A is pseudo-dense in X by Lemma 5.19. Thus we get statement (1).

Now suppose $X^1_\alpha, \dots, X^n_\alpha$ and A' are as stated in (2). Since A' is a subset of A, $\mathcal{D}_\beta(A', X)$ is a subset of $\mathcal{D}_\beta(A, X)$ for all β. So in particular, $\mathcal{D}_\beta(A', X) = \emptyset$ for all $\alpha < \beta < \dim X$. Suppose X_α is an element of $\mathcal{D}_\alpha(A', X)$. Then A is also pseudo-dense in X_α and so X_α is almost equal to X^i_α with $i \in \{1, \dots, n\}$. As $X^i_\alpha \cap A' = \emptyset$, X^i_α and X_α are both almost irreducible, and $\dim X^i_\alpha = \dim X_\alpha$, it follows from Lemma 5.1 that A' is not pseudo-dense in X_α which is absurd. Thus,

$$\mathcal{D}_\alpha(A', X) = \emptyset \quad \text{for all } \alpha \le \beta < \dim X.$$

If A' is pseudo-dense in X then clearly A is. Suppose A' is not pseudo-dense in X. Then $A' \cap X$ has a pseudo-closure Y' with $\dim Y' < \dim X$. It follows that A has a pseudo-closure Y which is a subset of $Y' \cup X^1_\alpha \cup \ldots \cup X^n_\alpha$. It is easy to see that $\dim Y < \dim X$, and so A is not pseudo-dense in X. We have thus obtained all the desired conclusions in (2). □

The above lemma is hardly useful if the purpose is defining pseudo-denseness for a general $\aleph_0$-stable theory. The issue is that many of the objects involved in the previous lemma are not definable. Remarkably, many of them are because T is $\aleph_0$-stable and $\aleph_0$-categorical. We recall a number of facts about $\aleph_0$-stable and $\aleph_0$-categorical structures.

Fact 5.33. The first two statements below only require $\aleph_0$-categoricity:

1. T is complete,
2. For all finite x, there are finitely many formula $\varphi(x)$ up to T equivalence;
3. T defines multiplicity;
4. ([CHL85], Theorem 5.1) $\mathcal{M}$ has finite Morley rank, that is, for all finite x, $\dim M^x < \omega$.
5. ([CHL85], Theorem 6.3) if x is a single variable, and $p \in S^x(\mathcal{M})$, then p is definable over $M^x \times M^x$.

We now prove a key lemma that does not hold outside of the $\aleph_0$-categorical setting.

Lemma 5.34. *For each finite x there is an L-formula $\psi(x,z)$ such that whenever $\mathcal{M} \models T$ and $\mathcal{D} = (X_c)_{c\in Z}$ is the family of subsets of M^x defined by $\psi(x,z)$, we have that every member of $\mathcal{D}$ is almost irreducible and every almost irreducible subset of M^x is almost equal to a member of $\mathcal{D}$.*

Proof. Fix $\mathcal{M} \models T$ of the given T, and a finite tuple x of variables. We reduce the problem to finding a formula $\psi(x,z)$ independent of the choice of $\mathcal{M}$ such that with $\mathcal{D} = (X_c)_{c\in Z}$ the family of subsets of M^x defined by $\psi(x,z)$, every almost irreducible X is almost equal to X_c for some $c \in M^z$. The analogous statement also hold in other models of T as T is complete. As T defines multiplicity, every almost irreducible $X \subseteq M^x$ has mult$X = 1$ and allow us to modify $\psi(x,z)$ suitably to get a formula as prescribed in the statement of the lemma.

We reduce the problem further to showing that every almost irreducible $X \subseteq M^x$ is almost equal to a subset of M^x which is $\mathcal{M}$-definable over some element of M^w with $|w| = 2|x|$. Suppose we have done so. By Fact 5.33(2), there are finitely many formulas $\psi_1(x,w),\dots,\psi_l(x,w)$ such that every L-formula in variables (x,w) is T-equivalent to one of these. By routine manipulation, we can get a finite tuple z of L-variables and a formula $\psi(x,z)$ such that for all $i \in \{1,\dots,l\}$ and $d \in M^w$, there is $c \in M^z$ with $\psi_i(M^x,d) = \psi(M^x,c)$. Hence, we obtained the desired reduction.

Let $p \in S^x(M)$ be the generic type of X and p^{eq} the unique element of $S^x(M^{\text{eq}})$ extending p. By merging the sorts, we can arrange that $|x| = 1$. By Fact 5.33(5), there is $c \in M^2$ such that p is definable over c. Hence p^{eq} is definable over c and therefore stationary over $\text{acl}^{\text{eq}}(c)$. It follows that

$$q = p^{\text{eq}} \upharpoonright S^x(\text{acl}^{\text{eq}}(c)) \text{ has mult} q = 1.$$

Let $X' \subseteq M^x$ be defined by a minimal formula of q. Then X' is $\mathcal{M}^{\text{eq}}$-definable over $\text{acl}^{\text{eq}}(b)$ and X' is almost equal to X. Let $X'_1,\dots,X'_l$ be all the finitely many conjugates of X' by $\text{Aut}(\mathcal{M}/c)$. Then $\bigcap_{i=1}^{l} X'_i$ is $\mathcal{M}$-definable over c and is almost equal to X which is the desired conclusion. □

A **function up-to-permutation** from $Z \subseteq M^z$ to M^w is a relation $f \subseteq Z \times M^w$ satisfying the following two conditions:

(1) for all $c \in Z$, there is $d \in M^w$ such that $(c,d) \in f$
(2) if (c,d) and (c,d') are both in f, then d is a permutation of d'.

Hence, the data specified by f is the same as that of a usual function $\tilde{f} : Z \to M^w/\sim$ where $\sim$ is the equivalence relation defined by permutations. We are interested in f instead of such $\tilde{f}$ as it is possible that f is $\mathcal{M}$-definable while $\tilde{f}$ is $\mathcal{M}^{\text{eq}}$-definable but not $\mathcal{M}$-definable. For $C \subseteq Z$, we will write $f(Z)$ for the set

$$\{d \in M^w \mid \text{ there is } c \in C \text{ such that } (c,d) \in f\}.$$

It is easy to observe that $|f(Z)| = |w|!\tilde{f}(Z)$ with $\tilde{f}$ as above. In particular, $f(Z)$ is finite if and only if $\tilde{f}(Z)$ is.

The following fact only uses the assumption that T is complete and weakly eliminates imaginaries:

Fact 5.35. For all $\mathcal{M} \models T$, 0-definable $Z \subseteq M^z$, and 0-definable equivalence relation $R \subseteq Z^2$, there is w and a 0-definable function up-to-permutation from Z to M^w such that cRc' in Z if and only if $f(c) = f(c')$. Moreover, the choice of formula defining f can be made depending only on the choices of L-formulas defining Z and R but not on the choice of $\mathcal{M}$.

Proposition 5.36. *The theory T' defines pseudo-denseness over T if and only if T' eliminates $\exists^\infty$.*

Proof. For the forward direction, suppose T and T' are fixed, T' defines pseudo-denseness, $\varphi'(x,y)$ is an L'-formula, $\mathcal{M}' \models T'$, $\mathcal{M} = \mathcal{M}' \upharpoonright L$, $(X'_b)_{b\in Y'}$ is the family of subsets of M^x defined by $\varphi'(x,y)$. Our job is to show that the set of $b \in Y'$ with infinite X'_b can be defined by a formula whose choice might depend on $\varphi(x,y)$ but does not depend on $\mathcal{M}'$. Let $\mathcal{D} = (X_c)_{c\in Z}$ be family of subsets of M^x defined by an L-formula $\psi(x,z)$ as described in Lemma 5.34. If follows from Lemma 5.32 that X'_b is infinite if and only if there is $c \in Z$ such that

$$X'_b \text{ is pseudo-dense in } X_c \text{ and } \dim(X_c) > 0.$$

By assumption, the set of pairs (b,c) with X'_b pseudo-dense in X_c can be defined by a formula whose choice does not depend on $\mathcal{M}'$. By Fact 5.33, T defines multiplicity. In particular, the set of $c \in Z$ with $\dim X_c > 0$ can be defined by an L-formula whose choice does not depend on $\mathcal{M}'$. The desired conclusion follows.

For the backward implication, suppose T and T' are fixed, T' eliminates $\exists^\infty$, $\varphi'(x,y)$ and $\psi(x,z)$ are an L'-formula and an L-formula, $\mathcal{M}' \models T'$, $\mathcal{M} = \mathcal{M}' \upharpoonright L$, and $(X'_b)_{b\in Y'}$ and $(X_c)_{c\in Z}$ are the families of subsets of M^x defined by $\varphi'(x,y)$ and $\psi(x,z)$. Set

$$\mathfrak{Pd} = \{(b,c) \in M^{(y,z)} \mid X'_b \text{ is pseudo-dense in } X_c\}.$$

We need to show that $\mathfrak{Pd}$ can be defined by an L'-formula whose choice might depend on $\varphi'(x,y)$ and $\psi(x,z)$ but not on $\mathcal{M}'$.

We first reduce to the special case where $\psi(x,z)$ is a formula as described in Lemma 5.34. Let $\delta(x,w)$ be a formula as described in Lemma 5.34 and $(X_d)_{d\in W}$ the family of subsets of M^x defined by $\delta(x,w)$, and suppose we have proven the corresponding statement for $\delta(x,w)$. We note that X'_b is pseudo-dense in X_c for $b \in Y'$ and $c \in Z$ if and only if for all $d \in W$ with $\dim X_d = \dim X_c$, we have X'_d is pseudo-dense in X_d. The desired reduction follows from the special case and Fact 5.33 which states that T defines multiplicity.

We next make a further reduction. Note that by the reduction in the preceding paragraph $\mathcal{D} = (X_c)_{c\in Z}$ is a family as described in Lemma 5.32, so we will set ourselves up to apply this lemma. For $\alpha < \dim M^x$, $b \in Y$, and $c \in Z$, we define $\mathfrak{D}_{\alpha,b,c}$ to be the set of $d \in Z$ such that $\dim X_d = \alpha$, X'_b is pseudo-dense in X_d, and X_d is almost a subset of X_c. In other words, if $\mathcal{D}_\alpha(X'_b, X_c)$ is defined as in Lemma 5.32, then

$$d \text{ is in } \mathfrak{D}_{\alpha,b,c} \quad \text{if and only if} \quad X_d \text{ is in } \mathcal{D}_\alpha(X'_b, X_c).$$

Set $\mathfrak{Pd}^0$ to be the set of $(b,c) \in \mathfrak{Pd}$ with $\dim X_c = 0$. For $\alpha < \gamma \le \dim M^x$, set $\mathfrak{Pd}^\gamma = \{(b,c) \in \mathfrak{Pd} \mid \dim X_c = \gamma\}$ and set

$$\mathfrak{Pd}^\gamma_\alpha = \{(b,c) \in \mathfrak{Pd} \mid \dim X_c = \gamma \text{ and } \mathfrak{D}_{\beta,b,c} = \varnothing \text{ for all } \alpha < \beta < \gamma\}.$$

We reduce the problem further to showing $\mathfrak{Pd}^\gamma_\alpha$ can be defined by an L'-formula whose choice is independent of $\mathcal{M}'$ for all $\alpha < \gamma \le \dim M^x$. Note that $(b,c) \in M^{(y,z)}$ is in $\mathfrak{Pd}^0$ if an only if $X_c \subseteq X'_b$ and $\dim(X_c) = 0$, so $\mathfrak{Pd}^0$ can be defined by a formula whose choice is independent of $\mathcal{M}'$. Moreover, $\mathfrak{Pd} = \bigcup_{\beta<\dim M^x} \mathfrak{Pd}^\beta$ and $\mathfrak{Pd}^\beta = \mathfrak{Pd}^\beta_{\beta-1}$, so by Fact 5.33(4) we obtained the desired reduction.

We will show the statement in the previous paragraph by lexicographic induction on (γ,α). We first settle some simple cases. For $\gamma = 1$ and $\alpha = 0$, the condition $\mathfrak{D}_{\beta,b,c} = \varnothing$ for all $\alpha < \beta < \gamma$ is vacuous, the desired conclusion follows from the fact that T defines multiplicity and T' eliminates $\exists^\infty$. Suppose we have proven the statement for all smaller values of γ. It follows from $\mathfrak{Pd}^\beta = \mathfrak{Pd}^\beta_{\beta-1}$ for $\gamma > 0$ that $\mathfrak{Pd}^0, \dots, \mathfrak{Pd}^{\gamma-1}$ can be defined by L'-formulas whose choice is independent of $\mathcal{M}'$. Let

$$Z_\gamma = \{c \in Z \mid \dim(X_c) = \gamma\}.$$

Note for $\beta < \gamma$ and $(b,c) \in Y \times Z_\gamma$ that $d \in M^z$ is in $\mathfrak{D}_{\beta,b,c}$ if and only if $\dim X_d = \beta$ and $(b,d) \in \mathfrak{Pd}^\gamma_\beta$. Using the fact that T defines multiplicity, we get for each $\beta < \gamma$ that the family $(\mathfrak{D}_{\beta,b,c})_{(b,c)\in Y\times Z_\gamma}$ can be defined by a formula independent of the choice of $\mathcal{M}'$. We get from Lemma 5.32 that $(b,c) \in M^{(y,z)}$ is in $\mathfrak{Pd}^\gamma_0$ if and only if

$$\dim X_c = \gamma, \quad \mathfrak{D}_{\beta,b,c} = \varnothing \text{ for all } 0 < \beta < \gamma, \quad \text{ and } X'_b \text{ is infinite.}$$

Hence, $\mathfrak{Pd}^\gamma_0$ can be defined by an L'-formula independent of the choice of $\mathcal{M}'$ by the assumption that T' eliminates $\exists^\infty$ and Fact 5.33(3).

Suppose $0 < \alpha < \gamma \le \dim M^x$ and we have shown the statement for all lexicongraphic lesser values of (γ, α) not just for the formula $\varphi(x,y)$ but also for any similar chosen $\varphi^*(x,y^*)$. From the argument in the preceding paragraph, $\mathfrak{Pd}^0, \dots, \mathfrak{Pd}^{\gamma-1}$ and $(\mathfrak{D}_{\beta,b,d})_{(b,c)\in Y\times Z_\gamma}$ for each $\beta < \gamma$ can be defined by formulas independent of the choice of $\mathcal{M}'$. By the assumption that T weakly eliminates $\exists^\infty$ and Fact 5.35, there is w and a L-definable function up-to-permutation f from Z to M^w defined by a formula whose choice does not depend on $\mathcal{M}'$ such that for all d_1 and d_2 in Z,

$$f(d_1) = f(d_2) \text{ if and only if } X_{d_1} \text{ is almost equal to } X_{d_2}.$$

In particular, the family $(f(\mathfrak{D}_{\alpha,b,c}))_{(b,c)\in Y\times Z_\gamma}$ can be defined by a formula whose choice does not depend on $\mathcal{M}'$. As T' eliminates $\exists^\infty$, there is n such that

$$|f(\mathfrak{D}_{\alpha,b,c})| > n|w|! \text{ implies } f(\mathfrak{D}_{\alpha,b,c}) \text{ is infinite.}$$

Now let Y^* be the set of $b^* = (b,c,d_1,\dots,d_n)$ in $Y \times Z \times \dots \times Z$ where the product by Z is taken $n+1$-times such that the following properties hold:

(1) $c \in Z_\gamma$ and $\mathfrak{D}_{\beta,b,c} = \varnothing$ for all $0 < \beta < \gamma$;
(2) $f(\mathfrak{D}_{\alpha,b,c})$ is finite;
(3) $\dim X_{d_i} = \alpha$ and X'_b is pseudo-dense in Z_{d_i} for $i \in \{1,\dots,n\}$;
(4) If $\dim X_d = \alpha$ and X'_b is pseudo-dense in Z_d for some $d \in Z$, then X_d is almost equal to X_{d_i} for some $i \in \{1,\dots,n\}$.

For each $b^* \in Y^*$, set

$$X'_{b^*} = X'_b \setminus \bigcup_{i=1}^n X_{d_i}.$$

Then by the induction hypothesis and Fact 5.33(3) the family $(X'_{b^*})_{b^*\in Y^*}$ can be defined by a formula $\varphi^*(x,y^*)$ whose choice does not depend on $\mathcal{M}'$. We obtain $\mathfrak{Pd}^{*\gamma}_{\alpha-1}$ from $\varphi^*(x,y^*)$ in the same fashion as we get $\mathfrak{Pd}^\gamma_{\alpha-1}$ from $\varphi(x,y)$. The induction hypothesis implies that $\mathfrak{Pd}^{*\gamma}_{\alpha-1}$ can be defined by formulas whose choice does not depend on $\mathcal{M}'$. It follows from Lemma 5.32 that $(b,c) \in \mathfrak{Pd}^\gamma_\alpha$ if and only if $\dim Z_c = \gamma$ and $\mathfrak{D}_{\beta,b,c} = \varnothing$ for all $\alpha < \beta < \gamma$ and either of the following hold:

(1) $f(\mathfrak{D}_{\alpha,b,c})$ is infinite;
(2) there are $d_1,\dots,d_n$ in Z such that $b^* = (b,c,d_1,\dots,d_n)$ is in Y^* and

$$X'_{b^*} \text{ is in } \mathfrak{Pd}^{*\gamma}_{\alpha-1}.$$

Thus $\mathfrak{Pd}^\gamma_\alpha$ can be defined by a formula whose choice does not depend on $\mathcal{M}'$ which completes the proof. ☐

Theorem 5.37. *Suppose L has finitely many sorts, $T_\cap$ is an $\aleph_0$-stable and $\aleph_0$-categorial theory which weakly eliminates imaginaries, and T_i eliminates $\exists^\infty$ for all $i \in I$. Then $T^*_\cup$ exists.*

The theory T_p of vector spaces over the finite field $\mathbb{F}_p$ with p elements is $\aleph_0$-stable, $\aleph_0$-categorical, and weakly eliminates imaginaries. Thus any theory T' extending T_p defines pseudo-denseness if and only if it eliminates $\exists^\infty$. This does not generalize to vector spaces over characteristic zero fields (which are $\aleph_0$-stable and weakly eliminate imaginaries, but are not $\aleph_0$-categorical). Let T be the theory of divisible abelian groups. Let T' be any theory of characteristic zero fields extending T. Then T' does not define pseudo-denseness over T. Suppose $\mathcal{M}'$ is an $\aleph_1$-saturated model of T'. Let

$$L = \{(a,b,c) \in \mathbf{M}^3 : ab = c\}$$

and consider the definable family $\{L_a : a \in M\}$ where $L_a = \{(b,c) \in \mathbf{M}^2 : ab = c\}$. We leave the easy verification of the following to the reader:

Lemma 5.38. *Fix $a \in \mathbf{M}$. Then L_a is pseudo-dense in $\mathbf{M}^2$ if and only if $a \in \mathbb{Q}$.*

As $\mathbb{Q}$ is countable and infinite it cannot be a definable set in an $\aleph_1$-saturated structure. Thus $\mathcal{M}'$ does not define pseudo-denseness over $(\mathbf{M};+)$.

There is a natural rank rk on any $\aleph_0$-categorical theory, described in [Sim18b, Section 2.3] and [CH03, Section 2.2.1]. This rank is known to agree with thorn rank on $\aleph_0$-categorical structures, so it is an elementary rank on rosy $\aleph_0$-categorical theories. A special case of Theorem 5.18 is that any expansion of the theory DLO of dense linear orders defines psuedo-denseness over DLO with respect to this elementary rank (which agrees with the usual o-minimal dimension over DLO). This fact, together with Proposition 5.36, and recent groundbreaking work on NIP $\aleph_0$-categorical structures [Sim18b, Sim18a] motivates the following question:

Question 5.39. Suppose T is NIP, $\aleph_0$-categorical, and rosy. If T' eliminates $\exists^\infty$ than must T define psuedo-denseness over T (with respect to rk)?

Unfortunately rk does not necessarily agree with Morley rank on $\aleph_0$-stable, $\aleph_0$-categorical theories. One might hope that an approach to Question 5.39 would synthesis the ideas of Section 5.5 and Section 5.3.

6. Examples

6.1. **Winkler's thesis.** Suppose L is $\{=\}$ and T is the theory of an infinite set with equality. Winkler proves Proposition 6.1 below in his thesis [Win75]. As T is $\aleph_0$-categorical, $\aleph_0$-stable, and weakly eliminates imaginaries, Proposition 6.1 is a special case of Theorem 5.37.

Proposition 6.1. *Suppose each T_i eliminates $\exists^\infty$. Then $T^*_\cup$ exists.*

Proposition 6.1, Proposition 4.13, the fact that a theory with trivial algebraic closure eliminates $\exists^\infty$, and Proposition 4.14, together imply Corollary 6.2.

Corollary 6.2. *Suppose* acl_i *is trivial for all $i \in I$. Then $T^*_\cup$ exists. If $\mathcal{M}_\cup \models T^*_\cup$ then every $\mathcal{M}_\cup$-definable set is a Boolean combination of $\mathcal{M}_i$-definable sets for various $i \in I$. If each T_i is additionally stable (NIP) then $T^*_\cup$ is stable (NIP).*

The special case of Corollary 6.2 when T_2 is the theory of dense linear orders is proven in [SS12, Corollary 1.2].

6.1.1. *The Expansion by a Generic Predicate.* Suppose $I = \{1,2\}$ and T_1 eliminates $\exists^\infty$. Suppose T_2 is the theory of an infinite set equipped with an infinite and co-infinite subset P. Note T_2 is $\aleph_0$-categorical and hence eliminates $\exists^\infty$. So Proposition 6.1 implies $T^*_\cup$ exists. This theory is known as the expansion of T_1 by a generic unary predicate, defined by Chatzidakas and Pillay [CP98] [DMS13].

Let $\mathcal{M}_1 \models T_1$. Fix a subset P of M. Let $\mathcal{M}_2$ be the T_2-model $(M;P)$ and $\mathcal{M}_\cup$ be the expansion of $\mathcal{M}_1$ by a predicate defining P. We recall the usual definition of genericity for P. An $\mathcal{M}_1$-definable set $X_1 \subseteq M^n$ is said to be *large* if there is an element $(a_1,\ldots,a_n)$ of $X_1(\boldsymbol{\mathcal{M}}_1)$ such that $a_i \notin M$ and $a_i \neq a_j$ for all $i \neq j$. Then P is generic if and only if the following holds: for every large $\mathcal{M}_1$-definable $X_1 \subseteq M^n$ and $S \subseteq \{1,\ldots,n\}$ there is an $(a_1,\ldots,a_n) \in X_1$ such that $a_k \in P$ if and only if $k \in S$ for all $1 \le k \le n$. Equivalently, if every large $\mathcal{M}_1$-definable subset of M^n intersects every subset of the form $S_1 \times \ldots \times S_n$ where $S_i \in \{P, M \smallsetminus P\}$ for $1 \le i \le n$.

It is shown in [CP98] that P is generic if and only if $\mathcal{M}_\cup$ is a model of $T^*_\cup$. It follows that P is generic if and only if $\mathcal{M}_\cup$ satisfies the pseudo-topological axioms. We describe how one may directly derive this equivalence modulo several small facts which we leave to the reader.

We let adim be algebraic dimension on T as defined in Section 5.4.1. We let adim$'$ be the ordinal rank on T_1, T_2 induced by adim. As algebraic closure in T is trivial, it follows directly from the definitions that an $\mathcal{M}_1$-definable subset X_1 of M^n is large if and only if $\mathrm{adim}'(X_1) = n$. As $\mathrm{adim}'(X_1) < n$ if and only if X_1 is contained in an $(M;=)$-definable set of Morley rank $< n$, and M^n has Morley degree one (as an $(M;=)$-definable set), it follows by Lemma 5.19 that X_1 is psuedo-dense in M^n if and only if $\mathrm{adim}'(X_1) = n$. The following fact may be proven using quantifier elimination for T_2.

Fact 6.3. If $X_2 \subseteq M^n$ is $\mathcal{M}_2$-definable and $\mathrm{adim}' X_2 = n$ then X_2 contains a set of the form

$$\left(\prod_{i=1}^n S_i\right) \smallsetminus \left\{(a_1,\ldots,a_n) \in M^k : a_i = a_j \text{ for some } 1 \le i < j \le n\right\}$$

where $S_i \in \{P, M \smallsetminus P\}$ for all $1 \le i \le n$.

Note that again any $\mathcal{M}_2$-definable subset X_2 of M^n is pseudo-dense in M^n if and only if $\mathrm{adim}'(X_2) = n$. It follows that P is generic if and only if $X_1 \cap X_2 \neq \varnothing$ whenever $X_1 \subseteq M^k$ is $\mathcal{M}_1$-definable, $X_2 \subseteq M^k$ is $\mathcal{M}_2$-definable, and both X_1 and X_2 are pseudo-dense in M^k. In other words, P is generic if and only if $\mathcal{M}_\cup$ is $(M^k)_{n\in\mathbb{N}}$-approximably interpolative.

Now apply Fact 6.4, which may be proven using quantifier elimination for T.

Fact 6.4. $(M^k)_{k\in\mathbb{N}}$ is a pseudo-cell collection for $(M;=)$.

6.1.2. *Generic Skolemizations.* Throughout this section L is one-sorted and T is a model complete consistent L-theory eliminating $\exists^\infty$. Proposition 6.5 is proven in [Win75]:

Proposition 6.5. *There is a language L_{Skolem} extending L and a consistent L_{Skolem}-theory T_{Skolem} extending T such that T_{Skolem} is model complete and admits definable Skolem functions.*

It is possible to recover Proposition 6.5 as an application of Theorem 5.37. In this subsection we describe the simplest interesting case of the necessary construction. Suppose φ is a two-ary L-formula such that

$$T \models \forall x \exists^{=\infty} y \varphi(x, y).$$

Let f be a unary function symbol not in L and declare

$$T_+ = T \cup \{\forall x \varphi(x, f(x))\}.$$

We sketch a proof of the following:

Fact 6.6. T_+ has a model companion.

Proof. (Sketch) Suppose T_0 is the two sorted theory such that $(E_1, E_2; \pi) \models T$ when E_1 and E_2 are infinite sets and π is a surjection $E_2 \to E_1$ such that $\pi^{-1}(a)$ is infinite for all $a \in E_1$. It is easy to see that T_0 is $\aleph_0$-categorical and $\forall\exists$. Lindstorms test [Hod93, 8.3.4] shows that T_2 is model complete. If E is an infinite set then $(E, E^2; \rho) \models T$ when $\rho(a, b) = a$ for all $(a, b) \in E^2$. Thus T_0 is interpretable in the theory of an infinite set with equality, hence T_0 is $\aleph_0$-stable. It is also easy to see that T_0 admits weak elimination of imaginaries.

Let T_1 the two-sorted theory such that $(\mathcal{M}, E_2; \pi, \sigma) \models T$ when $\mathcal{M}$ is a T-model, E_2 is an infinite set, σ is a bijection between $\varphi(M^2)$ and E_2 such that $(\pi \circ \sigma)(a, b) = a$ for all $(a, b) \in M^2$ such that $\mathcal{M} \models \varphi(a, b)$. There is an existential bi-interpretation between T_1 and T, so T_1 is model complete and eliminates $\exists^\infty$.

Let T_2 be the two-sorted theory such that $(E_1, E_2; \pi, f) \models T_2$ when $(E_1, E_2; \pi) \models T_2$ and $f : E_1 \to E_2$ satisfies $(\pi \circ f)(a) = a$ for all $a \in E_1$. It is easy to see that T_2 is $\aleph_0$-categorical and thus eliminates $\exists^\infty$. It is also easy to see that T_2 is $\forall\exists$, hence model complete by Lindstorms test.

Theorem 5.37 shows $T_\cup$ has a model companion. It is easy to construct an existential bi-interpretation between $T_\cup$ and T_+. It follows by Corollary 2.18 that T_+ has a model companion. □

Suppose T is the theory of an infinite set with equality. If φ is $\top$ then T_+ is the theory of a unary function f from an infinite set to itself and the model companion of T_+ is the model companion of the theory of a function from a set to itself. It follows from the proof of Fact 6.6 that in this case T_+ is existentially bi-interpretable with a union of two theories, each of which is mutually existentially interpretable with the theory of an infinite set with equality. Thus the model companion of the theory of set equipped with a self-map is bi-interpretable with the interpolative fusion of two theories, each of which is mutually interpretable with the theory of an infinite set with equality. A slight modification of the proof of Fact 6.6 shows that the same claim holds for the model companion of the theory of a set X equipped with a function $f : X^n \to X$.

6.2. **The random graph.** Let L be the language of graphs with edge relation R. Let T be the L-theory of (undirected, loopless) infinite graphs with infinitely many edges. Then the model companion of T is the theory of the random graph. We show that T is existentially bi-interpretable with a union of two model complete theories, each of which is mutually existentially interpretable with the theory of an infinite set with equality. An application of Corollary 2.18 shows that the theory of the random graph is bi-interpretable with the interpolative fusion of two theories, each of which is mutually interpretable with the theory of an infinite set with equality.

Given a set X we let

$$\Delta(X) = \{(x,y) \in X^2 : x \neq y\},$$

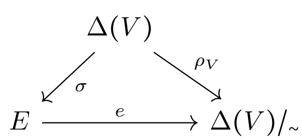
let $\sim$ be the equivalence relation on $\Delta(X)$ given by permuting coordinates, and let $\rho_X : \Delta(X) \to \Delta(X)/_\sim$ be the quotient map.

Let $T_\cap$ be the two sorted theory of two infinite sets V, E. Let L_1 be the expansion of the language of $T_\cap$ by σ, and let T_1 be the L_1-theory such that $(V, E; \sigma) \models T_1$ if V, E are infinite sets and σ is a surjection $\Delta(V) \to E$ such that $\sigma(a) = \sigma(b)$ if and only if $a \sim b$. Let L_2 be the expansion of the language of $T_\cap$ by a unary predicate P and T_2 be the L_2-theory such that $(V, E; P) \models T_2$ when V, E are infinite sets and P is an infinite and co-infinite subset of E. Then $T_\cap, T_1$, and T_2 are easily seen to be mutually existentially interpretable with the theory of an infinite set.

Proposition 6.7. *There is an existential bi-interpretation (F, G, η, η') between T and $T_\cup$.*

Proof. (Sketch) Suppose $(X; R)$ is a T-model. Let $Q \subseteq \Delta(X)/_\sim$ be the image of R under ρ_X. Then $F(X; R)$ is $(X, \Delta(X)/_\sim; \rho_X, Q)$. Suppose $H = (V, E; \sigma, P) \models T_\cup$. Then $G(H)$ is $(V; R)$ where R is the preimage of P under ρ.

As $G(F(X; R)) = (X; R)$ we let $\eta_{(X;R)}$ be the identity. Note $F(G(H))$ is $(V, \Delta(V)/_\sim; \rho_V, Q)$. Let $e : E \to \Delta(V)/_\sim$ be the unique bijection such that



commutes. Then (id_V, e) gives an isomorphism between H and $F(G(H))$ so we let η'_H be (id_V, e). □

Note that $T_\cap$ is $\aleph_0$-stable, $\aleph_0$-categorical, and has weak elimination of imaginaries. Note also that T_1 and T_2 are $\aleph_0$-categorical and hence eliminate $\exists^\infty$. Thus the well-known fact that T has a model companion may be seen as a very special case of Theorem 5.37. Furthermore, note that algebraic closure in T_2 is trivial and thus agrees with algebraic closure in $T_\cap$. Thus, the well-known fact that the random graph is $\aleph_0$-categorical may be seen as a special case of the theorem of Pillay and Tsuboi [PT97] discussed in Section 4.6.

One can easily modify this example to show that the random n-hypergraph, the random directed graph, and the random bipartite graph, can each be expressed as the interpolative fusion of two theories, each of which is bi-interpretable with the theory of an infinite set with equality.

6.3. **Generic automorphisms and ACFA.** In this subsection T is a one-sorted model-complete consistent L-theory. Let L_σ be the extension of L by a new unary function symbol σ and T_σ be the theory such that $(\mathcal{M}, \sigma) \models T_\sigma$ if and only if $\mathcal{M} \models T$ and σ is an automorphism of $\mathcal{M}$.

The existence of a model companion of T_σ is tied to classification-theoretic issues. If T has the strict order property then T_σ does not have a model companion [KS02]. It is conjectured that if T is unstable then T_σ does not have a model companion. Baldwin and Shelah [BS01] gave necessary and sufficient conditions for T_σ to admit a model companion when T is stable.

We construct an existential bi-interpretation between T_σ and a union of two theories each of which is existentially bi-interpretable with T. The problem of finding a model companion of T_σ is thus a special case of the problem of finding a model companion of a union of two theories.

Suppose L_1, L_2 are one-sorted first order languages. Let $L_1 \sqcup L_2$ be the disjoint union of L_1 and L_2. The set of sorts of $L_1 \sqcup L_2$ is the disjoint union of the set of sorts of L_1 and the set of sorts of L_2. Given L_1, L_2-structures $\mathcal{M}_1, \mathcal{M}_2$ with domains M_1, M_2 we define the disjoint union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ to be an $L_1 \sqcup L_2$-structure with domain $M_1 \sqcup M_2$ in the natural way. Any $L_1 \sqcup L_2$-structure is the disjoint union of an L_1-structure and an L_2-structure.

Suppose $\mathcal{M}_1 \sqcup \mathcal{M}_2$ and $\mathcal{N}_1 \sqcup \mathcal{N}_2$ are $L_1 \sqcup L_2$-structures. If $\mathcal{M}_1 \xrightarrow{g_1} \mathcal{N}_1$ and $\mathcal{M}_2 \xrightarrow{g_2} \mathcal{N}_2$ are isomorphisms then $g_1 \sqcup g_2$ gives an isomorphism $\mathcal{M}_1 \sqcup \mathcal{M}_2 \to \mathcal{N}_1 \sqcup \mathcal{N}_2$. Conversely, any isomorphism between $L_1 \sqcup L_2$ structures decomposes as a disjoint union of two isomorphisms in this way. It is easy to see that Proposition 6.8 below is true.

Proposition 6.8. *Suppose $\mathcal{M}_1 \sqcup \mathcal{M}_2$ is an $L_1 \sqcup L_2$ structure. Any $\mathcal{M}_1 \sqcup \mathcal{M}_2$-definable subset of $(M_1)^x \times (M_2)^y$ is a finite union of sets of the form $A \times B$ for $\mathcal{M}_1$-definable $A \subseteq (M_1)^x$ and $\mathcal{M}_2$-definable $B \subseteq (M_2)^y$.*

Fact 6.9 follows from Proposition 6.8. We leave the proof as an exercise to the reader.

Fact 6.9. The disjoint union of two theories weakly eliminating imaginaries also weakly eliminates imaginaries.

For the rest of the subsection, let I be $\{1,2\}$, $L_\cap$ be $L \sqcup L$, $T_\cap = T \sqcup T$, L_i extends $L_\cap$ by a function symbol τ_i from the first sort to the second sort, and T_i is the L_i theory such that $(\mathcal{M}, \mathcal{N}; \tau_i) \models T_i$ if and only if

$$\mathcal{M}, \mathcal{N} \models T, \quad \text{and} \quad \mathcal{M} \xrightarrow{\tau_i} \mathcal{N} \text{ is an } L\text{-isomorphism },$$

for $i \in \{1,2\}$.

Any isomorphism between two T_1-models $(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$ and $(\mathcal{M}^* \sqcup \mathcal{N}^*; \tau_1^*)$ is of the form $g \sqcup h$ for L-isomorphisms $\mathcal{M} \xrightarrow{g} \mathcal{M}^*$ and $\mathcal{N} \xrightarrow{h} \mathcal{N}^*$ such that

$$\begin{array}{ccc} M & \xrightarrow{\tau_1} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ M^* & \xrightarrow{\tau_1^*} & N^* \end{array}$$

commutes.

Fact 6.10. There is an existential bi-interpretation (F, G, η, η') between T_1 and T.

Proof. Let $F(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$ be the L-reduct on the first sort, i.e. $\mathcal{M}$. Suppose $\mathcal{M}$ is a T-model and let $\mathrm{id}_\mathcal{M}$ be the identity on $\mathcal{M}$. Let $G(\mathcal{M})$ be $(\mathcal{M} \sqcup \mathcal{M}; \mathrm{id}_\mathcal{M})$. Then $G(\mathcal{M})$ is a quantifier-free $\mathcal{M}$-definable T_1-model.

We let $\eta_\mathcal{M}$ be the identity map $\mathcal{M}$, this is an $\mathcal{M}$-definable L-isomorphism between $\mathcal{M}$ and $F(G(\mathcal{M}))$. In the other direction, fix a T_1-model $\mathcal{R} = (\mathcal{M} \sqcup \mathcal{N}, \tau_1)$. Then

$G(F(\mathcal{R}))$ is $(\mathcal{M} \sqcup \mathcal{M}; \mathrm{id}_{\mathcal{M}})$. The diagram

$$\begin{array}{ccc} M & \xrightarrow{\tau_1} & N \\ {\scriptstyle \mathrm{id}_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \tau_1^{-1}} \\ M & \xrightarrow{\mathrm{id}_{\mathcal{M}}} & M \end{array}$$

commutes, so $\eta'_R := \mathrm{id}_{\mathcal{M}} \sqcup \tau_1^{-1}$ is an $(L \sqcup L, \tau)$-isomorphism between $(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$ and $(\mathcal{M} \sqcup \mathcal{M}; \mathrm{id}_{\mathcal{M}})$. Note $\mathrm{id}_{\mathcal{M}} \sqcup \tau_1^{-1}$ is $(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$-definable. □

Now $T_\cup$ is the theory of a $T \sqcup T$-model $\mathcal{M} \sqcup \mathcal{N}$ together with two L-isomorphisms $\tau_1, \tau_2 : \mathcal{M} \to \mathcal{N}$. Any isomorphism between two $T_\cup$-models $(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ and $(\mathcal{M}^* \sqcup \mathcal{N}^*; \tau_1^*, \tau_2^*)$ is of the form $g \sqcup h$ for L-isomorphisms $\mathcal{M} \xrightarrow{g} \mathcal{N}$ and $\mathcal{M}^* \xrightarrow{h} \mathcal{N}^*$ such that both

$$\begin{array}{ccc} M & \xrightarrow{\tau_1} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ M^* & \xrightarrow{\tau_1^*} & N^* \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{\tau_2} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ M^* & \xrightarrow{\tau_2^*} & N^* \end{array}$$

commute.

Fact 6.11. There is an existential bi-intrepretation (F, G, η, η') between $T_\cup$ and T_σ.

Proof. Suppose $(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ is a $T_\cup$-model. Then $\tau_1^{-1} \circ \tau_2$ is an L-isomorphism of $\mathcal{M}$. We let $F'(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ be $(\mathcal{M}; \tau_1^{-1} \circ \tau_2)$. Suppose $(\mathcal{M}; \sigma) \models T_\sigma$. Then we declare $G'(\mathcal{M}; \sigma)$ to be $(\mathcal{M} \sqcup \mathcal{M}; \mathrm{id}_{\mathcal{M}}, \sigma)$. This gives a Δ_1-interpretation as $M \sqcup M, \mathrm{id}_{\mathcal{M}}$, and σ are quantifier free $(\mathcal{M}; \sigma)$-definable. It follows directly from the definitions that $F(G(\mathcal{M}; \sigma))$ is $(\mathcal{M}; \sigma)$ so we take $\eta_{(\mathcal{M};\sigma)}$ to be $\mathrm{id}_{\mathcal{M}}$.

Suppose $\mathcal{R} = (\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ is a $T_\cup$-model. Then $G(F(\mathcal{R}))$ is $(\mathcal{M} \sqcup \mathcal{M}; \mathrm{id}_{\mathcal{M}}, \tau_1^{-1} \circ \tau_2)$. Both

$$\begin{array}{ccc} M & \xrightarrow{\tau_1} & N \\ {\scriptstyle \mathrm{id}_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \tau_1^{-1}} \\ M & \xrightarrow{\mathrm{id}_{\mathcal{M}}} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{\tau_2} & N \\ {\scriptstyle \mathrm{id}_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \tau_1^{-1}} \\ M & \xrightarrow{\tau_1^{-1} \circ \tau_2} & M \end{array}$$

commute so $\eta'_{\mathcal{R}} := \mathrm{id}_{\mathcal{M}} \sqcup \tau_1^{-1}$ gives an existentially $(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$-definable $L_\cup$-isomorphism

$$(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2) \to (\mathcal{M} \sqcup \mathcal{M}; \mathrm{id}_{\mathcal{M}}, \tau_1^{-1} \circ \tau_2).$$

□

Applying Corollary 2.18 and the easy fact that T_σ is inductive we obtain:

Proposition 6.12. *T_σ has a model companion if and only if $T_\cup$ has a model companion, and the two model companions are existentially bi-interpretable whenever they exist.*

We also obtain the following:

Proposition 6.13. *Suppose T is $\aleph_0$-stable and $\aleph_0$-categorical with weak elimination of imaginaries. Then T_σ has a model companion.*

Proof. As T_σ and $T_\cup$ are existentially bi-interpretable it suffices to show $T_\cup$ has a model companion. As any countable model of $T \sqcup T$ is a disjoint union of two countable models of T, $\aleph_0$-categoricity of T implies $T \sqcup T$ is $\aleph_0$-categorical. Furthermore $T \sqcup T$ is $\aleph_0$-stable as it is interpetable in T. Fact 6.9 shows $T \sqcup T$ weakly eliminates imaginaries.

As T is $\aleph_0$-categorical, it eliminates $\exists^\infty$. As T_1 and T_2 are definable in T, they also eliminate $\exists^\infty$. We may now apply Theorem 5.37 to conclude that $T^*_\cup$ exists. □

6.3.1. *ACFA.* We now consider the special case when T is ACF. The model companion of the theory of ACF_σ is known as ACFA. This well-known theory is treated in [CH99][Mac97] and many other places. Let $(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$ be a model of ACF_1, i.e. $\mathcal{M}$ and $\mathcal{N}$ are algebraically closed fields and $\mathcal{M} \xrightarrow{\tau_1} \mathcal{N}$ is a field isomorphism. The proof of Fact 6.10 implies that every $(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$-definable subset of $M^k \times N^l$ is of the form $\{(x, \tau_1(y)) : (x, y)) \in X\}$ for some $\mathcal{M}$-definable subset X of $M^k \times M^l$. We say that a definable subset of $M^k \times N^l$ is irreducible if its pre-image under $(x, y) \mapsto (x, \tau_1(y))$ is irreducible.

Lemma 6.14. *Suppose $V_1 \subseteq M^x, V_2 \subseteq N^y$ are irreducible subvarieties, π_1, π_2 are the coordinate projections from $V_1 \times V_2$ to V_1, V_2, and $X \subseteq V_1 \times V_2$ is an irreducible $(\mathcal{M} \sqcup \mathcal{N}, \tau_1)$-definable set. Then X is pseudo-dense in $V_1 \times V_2$ if and only if $\pi_i(X)$ is Zarski-dense in V_i for $i \in \{1, 2\}$.*

Proof. We prove both implications by contrapositive. Suppose $\pi_1(X)$ is not pseudo-dense in V_1. Fix a definable $\dim V_1$-dimensional $W \subseteq V_1$ disjoint from $\pi_1(X)$. Then X is disjoint from $W \times V_2$ and

$$\dim(W \times V_2) = \dim W + \dim V_2 = \dim V_1 + \dim V_2 = \dim(V_1 \times V_2).$$

So X is not pseudo-dense in $V_1 \times V_2$.

Suppose X is not pseudo-dense in $V_1 \times V_2$. Proposition 6.8 implies that there are definable $W_1 \subseteq V_1, W_2 \subseteq V_2$ such that $\dim(W_1 \times W_1) = \dim(V_1 \times V_2)$ and $W_1 \times W_2$ is disjoint from X. As

$$\dim W_1 \le \dim V_1, \dim W_2 \le \dim V_2 \quad \text{and} \quad \dim(W_1 \times W_2) = \dim W_1 + \dim W_2$$

we have $\dim W_1 = \dim V_1$ and $\dim W_2 = \dim V_2$. Let $Y_i = V_i \setminus W_i$ for $i \in \{1, 2\}$. As V_i is irreducible we have $\dim Y_i < \dim V_i$ for $i \in \{1, 2\}$. Letting Y'_i be the Zariski closure of Y_i we have $\dim Y'_i < \dim V_i$ for $i \in \{1, 2\}$. As X is disjoint from $W_1 \times W_2$ we have

$$X \subseteq (Y'_1 \times V_2) \cup (V_1 \times Y'_2).$$

As X is irreducible and $Y'_1 \times V_2, V_1 \times Y'_2$ are both Zariski-closed, we have

$$X \subseteq Y'_1 \times V_2 \quad \text{or} \quad X \subseteq V_1 \times Y'_2.$$

In the first case $\pi_1(X) \subseteq Y'_1$ and so $\pi_1(X)$ is not Zariski-dense in V_1. In the second case $\pi_2(X)$ is not Zariski-dense in V_2. □

Let σ be an automorphism of $\mathcal{M}$. Hrushovksi's *geometric axioms* [Mac97] state $(\mathcal{M}, \sigma) \models$ ACFA if and only if whenever V is an irreducible $\mathcal{M}$-variety, X is an irreducible subvariety of $V \times \sigma(V)$, and the projection of X onto $V, \sigma(V)$ is Zariski-dense in $V, \sigma(V)$, then $(a, \sigma(a)) \in X$ for some $a \in V$.

The pseudo-topological axioms obtained in this setting are logically equivalent to Hrushovski's geometric axioms as they both axiomitize the same class of structures.

In the remainder of this section we explain how Hrushovski's geometric axioms may be directly derived from our pseudo-topological axioms.

We describe the geometric axioms in our two-sorted setting. Suppose $(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ is a $T_\cup$-model and $\sigma = \tau_1^{-1} \circ \tau_2$. Then $(\mathcal{M}; \sigma)$ satisfies the geometric axioms when $(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ satisfies the following claim: If V is an irreducible $\mathcal{M}$-variety, X is an irreducible $(\mathcal{M} \sqcup \mathcal{N}; \tau_1)$-definable subset of $V \times \tau_2(V)$ such that the projection of X onto $V, \tau_2(V)$ is Zariski-dense in $V, \tau_2(V)$ respectively, then $(x, \tau_2(x)) \in X$ for some $x \in V$. Composing by $(x, y) \mapsto (x, \tau_1^{-1}(y))$ converts this claim into the geometric axioms.

Suppose V and X satisfy the conditions of the claim above. Suppose $(\mathcal{M} \sqcup \mathcal{N}; \tau_1, \tau_2)$ satisfies the pseudo-topological axioms. We wish to show that X and $\Gamma := \{(a, \tau_2(a)) : a \in V\}$ intersect. Lemma 6.14 implies X is pseudo-dense in $V \times \tau_2(V)$. It therefore suffices to show Γ is pseudo-dense in $V \times \tau_2(V)$. Applying Proposition 6.8 it is enough to suppose Y, Z are $\mathcal{M}, \mathcal{N}$-definable subsets of $V, \tau_2(V)$ such that $\dim Y = \dim V, \dim Z = \dim \tau_2(V)$ and show Γ intersects $Y \times Z$. As τ_2 is an isomorphism we have $\dim V = \dim \tau_2(V)$ and also

$$\dim \tau_2(Y) = \dim Y = \dim \tau_2(V).$$

As V is irreducible, $\tau_2(Y)$ intersects Z. Hence there is an $a \in Y$ such that $\tau_2(a) \in Z$. Thus $(a, \tau_2(a))$ is an element of both Γ and $Y \times Z$.

6.3.2. *Pseudofinite fields.* If $(K; \sigma) \models \mathrm{ACF}_\sigma$ then the fixed field of $(K; \sigma)$ is $\{a \in K : \sigma(a) = a\}$. A *pseudo-finite* field is an infinite field which is a model of the theory of finite fields. We recall a fact which is presumably well known to experts and easily follows from fundamental results on pseudofinite fields.

Fact 6.15. A field is pseudofinite if and only if it is elementarily equivalent to the fixed field of a model of ACFA.

Proof. See [Mac97, Theorem 6] for a proof of the fact that the fixed field of a model of ACFA is pseudofinite. Suppose K is a pseudofinite field. Let L be an algebraic closure of K and let σ be some automorphism of L with fixed field K. As ACFA is the model companion of the theory of fields equipped with an automorphism there is an ACFA-model $(L'; \sigma')$ such that $(L; \sigma)$ is a sub-difference field of $(L'; \sigma')$. Let F be the fixed field of $(L'; \sigma')$. Then K is a subfield of F, and the (field-theoretic) algebraic closure of K inside of F is equal to K. It follows that the algebraic closure of the prime subfield of F agrees with that of K. A well known theorem of Ax (see for example [Cha97, Theorem 1]) implies that K and F are elementarily equivalent. □

6.4. ***D*-rings and DCF.** In this section all rings are commutative with unit. We let 1_A be the unit of a ring A. We treat the D-ring formalism developed in [MS14]. Moosa and Scanlon work with D-rings defined over a general commutative base ring. To simplify exposition we only work with D-rings defined over the ring of integers. We leave many algebraic details to the reader.

The definition of a D-ring requires the following data: a ring D which is free and finitely generated as an additive group, a $\mathbb{Z}$-basis $\bar{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_m)$ of D, and a ring homomorphism $\pi \colon D \to \mathbb{Z}$ such that $\pi(\varepsilon_0) = 1$ and $\pi(\varepsilon_i) = 0$ for $i \geq 1$. As $\bar{\varepsilon}$ is a

$\mathbb{Z}$-basis there are integers $\{c_{ijk} \mid 0 \leq i,j,k \leq m\}$ satisfying

$$\varepsilon_i \varepsilon_j = \sum_{k=0}^{m} c_{ijk}\varepsilon_k \quad \text{for all } 0 \leq i,j \leq m$$

and integers $\{d_i \mid 0 \leq i \leq m\}$ such that

$$1_D = d_0\varepsilon_0 + \ldots + d_m\varepsilon_m.$$

As $\pi(1_D) = 1$ we must have $d_0 = 1$. A simple computation shows that multiplication in D is given by the following equation:

$$\left(\sum_{i=0}^{m} a_i\varepsilon_i\right)\left(\sum_{j=0}^{m} a'_j\varepsilon_j\right) = \sum_{k=0}^{m}\left(\sum_{i,j} c_{ijk}a_i a'_j\right)\varepsilon_k.$$

for all $a_0,\ldots,a_m,a'_0,\ldots,a'_m \in \mathbb{Z}$.

Let A be a domain. We define a ring $D(A)$ which is isomorphic to $D \otimes_{\mathbb{Z}} A$ and existentially definable in A: $D(A)$ has underlying set A^{m+1}, addition is given by

$$(a_0,\ldots,a_m) + (a'_0,\ldots,a'_m) = (a_0 + a'_0,\ldots,a_m + a'_m)$$

and multiplication is given by

$$(a_0,\ldots,a_m)(a'_0,\ldots,a'_m) = \left(\sum_{i,j} c_{ij0}a_i a'_j,\ldots,\sum_{i,j} c_{ijm}a_i a'_j\right)$$

for all $a_0,\ldots,a_m,a'_0,\ldots,a'_m \in A$. Note $D(A)$ has unit $\bar{d} = (d_0,\ldots,d_m)$. We let $\bar{u}$ be the standard basis for A^{m+1}. The map

$$(a_0,\ldots,a_m) \mapsto \sum_{i=0}^{m} a_i\varepsilon_i$$

gives a ring isomorphism $D(A) \to D \otimes_{\mathbb{Z}} A$ which takes u_i to ϵ_i for $0 \leq i \leq m$. Let $\pi_A : D(A) \to A$ be the projection onto the first coordinate, this is a ring homomorphism.

A D**-ring** is a domain A together with a sequence $\partial_1,\ldots,\partial_m$ of functions $A \to A$ such that

$$e(a) = (a,\partial_1(a),\ldots,\partial_m(a))$$

is a ring homomorphism $A \to D(A)$. Note that e is a section of π_A. To simplify notation we always declare $\partial_0(a) = a$ for all $a \in A$ so that

$$e(a) = \bar{\partial}(a) := (\partial_0(a),\ldots,\partial_m(a)).$$

Then $(A;\bar{\partial})$ is a D-ring if and only if ∂_0 is the identity, each ∂_i is additive, and

$$\bar{\partial}(aa') = \bar{\partial}(a)\bar{\partial}(a')$$

for all $a,a' \in A$, equivalently if

$$\partial_k(aa') = \sum_{i,j} c_{ijk}\partial_i(a)\partial_j(a')$$

for all $a,a' \in A$ and $0 \leq k \leq m$. Thus D-rings form an elementary class.

We construct an existential bi-interpretation between the theory of D-rings and a union of two theories, each of which is existentially bi-interpretable with the theory of domains. It is shown in [MS14] that the theory of D-rings admits a model companion. It follows by Corollary 2.18 that the model companion of the theory of D-rings is existentially bi-interpretable with a union of two theories, each of which is existentially bi-interpretable with ACF (the model companion of the theory of domains).

We describe two motivating examples. First let D be the ring $\mathbb{Z}[\varepsilon]/(\varepsilon^2)$ of dual numbers, let $\bar{\varepsilon}$ be $(1,\varepsilon)$, and let π be given by $\pi(l+k\varepsilon)=l$. Then $(a,b)\mapsto a+b\varepsilon$ gives a ring isomorphism between $D(A)$ and the ring $A[\varepsilon]/(\varepsilon^2)$ of dual numbers over A. Let $\delta\colon A\to A$ be a map. It is well known and follows by direct computation that the map $e: A\to A[\varepsilon]/(\varepsilon^2)$ given by $e(a)=a+\delta(a)\varepsilon$ is a homomorphism if and only if δ is a derivation. Thus a $\mathbb{Z}[\varepsilon]/(\varepsilon^2)$-ring is a differential ring and the model companion of the theory of $\mathbb{Z}[\varepsilon]/(\varepsilon^2)$-rings is the theory of differentially closed fields. Secondly, suppose D is $\mathbb{Z}\times\mathbb{Z}$, $\bar{\varepsilon}$ is $((1,0),(0,1))$, and $\pi(l,k)=l$. Then $D(A)$ is $A\times A$ (with the pointwise operations). If $\delta\colon A\to A$ is a map then $a\mapsto(a,\delta(a))$ gives a ring homomorphism $A\to A\times A$ if and only if δ is a ring homomorphism. Thus a $\mathbb{Z}\times\mathbb{Z}$-ring is simply a domain equipped with a self-homomorphism, i.e. a difference ring. The model companion of the theory of difference rings is ACFA.

For the remainder of this section we fix $D,\bar{\varepsilon}$, and π.

Given a domain A we consider the two-sorted structure $(D(A),A;\bar{u},\pi_A)$, here $D(A)$ and A are regarded as rings and $\bar{u}=(u_0,\dots,u_m)$ is the standard basis of A^{m+1}. We let T be the theory of this class of structures. Suppose $(Q,A;\bar{v},\rho)\models T$. It does not necessarily follow that Q is isomorphic as a ring to $D(A)$. Note however it does follows that

$$v_iv_j=\sum_{k=0}^{m}c_{ijk}v_k \quad \text{for all } 0\le i,j,\le m.$$

Lemma 6.16. *There is a mutual existential interpretation (E,F) between the theory of domains and T.*

Proof. We let $F(Q,A;\bar{v},\rho)$ be A and let $E(A)$ be $(D(A),A;\bar{u},\pi_A)$ as above. □

Let R be a ring. An A**-algebra** $\odot$ on R is an A-module structure $A\times R\to R$ on R such that $(a\odot r)(a'\odot r')=(aa')\odot(rr')$ for all $a,a'\in A$ and $r,r'\in R$. There is a canonical correspondence between A-algebras on R and ring homomorphisms $A\to R$. One associates a ring homomorphism τ to an A-algebra $\odot$ by declaring $\tau(a)=a\odot 1_R$ for all $a\in A$ and associates an A-algebra $\odot$ to a ring homomorphism τ by declaring $a\odot r=\tau(a)r$ for all $r\in R$ and $a\in A$.

Suppose $(Q,A;\bar{v},\rho)\models T$. We say that an A-algebra $\odot$ on Q is **compatible** if $\bar{v}$ is an A-linear basis for the resulting A-module structure on Q and ρ is an A-linear map, i.e. $\rho(a\odot r)=a\rho(r)$ for all $a\in A, r\in Q$. Note in particular that

$$\rho(a\odot 1_Q)=a\rho(1_Q)=a1_A=a.$$

So the homomorphism associated to a compatible A-algebra is a section of ρ. There is a canonical compatible A-algebra on $D(A)$ given by

$$a\odot(b_1,\dots,b_m)=(ab_1,\dots,ab_m).$$

We leave the details of Lemma 6.17 to the reader.

Lemma 6.17. *Suppose $\odot$ is a compatible A-algebra on Q. Then*

$$\left(\sum_{i=0}^{m}a_i\odot\varepsilon_i\right)\left(\sum_{j=0}^{m}a'_j\odot\varepsilon_j\right)=\sum_{k=0}^{m}\left(\sum_{i,j}c_{ijk}a_ia'_j\right)\odot\varepsilon_k$$

and

$$\rho\left(\sum_{i=0}^{m} a_i \odot v_i\right) = a_0$$

for all $a_0, \dots, a_m, a'_0, \dots, a'_m \in A$.

There is a canonical correspondence between

(1) isomorphisms $\vartheta\colon (D(A); \bar{u}, \pi_A) \to (Q; \bar{v}, \rho)$, and
(2) compatible A-algebras $\odot$ on Q.

In particular there is a compatible A-algebra on Q if and only if $(Q; \bar{v}, \rho)$ is isomorphic to $(D(A); \bar{u}, \pi_A)$. We associate a compatible A-algebra $\odot$ to an isomorphism $\vartheta\colon (D(A); \bar{u}, \pi_A) \to (Q; \bar{v}, \rho)$ by declaring

$$b \odot \vartheta(a_0 \dots, a_m) = \vartheta(ba_0, \dots, ba_m) \text{ for all } b, a_0, \dots, a_m \in A.$$

The next lemma describes how we associate an isomorphism to a compatible A-algebra. We leave the straightforward computational proof of Lemma 6.18 to the reader.

Lemma 6.18. *Suppose* $(Q, A; \bar{v}, \rho) \models T$. *If* $\odot$ *is a compatible* A*-algebra on* Q *then*

$$\vartheta(a_0, \dots, a_m) = \sum_{i=0}^{m} a_i \odot v_i$$

gives an isomorphism $(D(A); \bar{u}, \pi_A) \to (Q; \bar{v}, \rho)$.

The following lemma may be proven using Lemma 6.18 or by direct computation.

Lemma 6.19. *Suppose* $(Q, A; \bar{v}, \rho) \models T$ *and* $\odot$ *is a compatible* A*-algebra on* Q. *Let* $\bar{\partial} = \partial_0, \dots, \partial_m$ *be functions* $A \to A$. *Then* $(A; \bar{\partial})$ *is a* D*-ring if and only if* ∂_0 *is the identity and*

$$a \mapsto \sum_{i=0}^{m} \partial_i(a) \odot \varepsilon_i$$

is a ring homomorphism $A \to Q$.

We let $T_\odot$ be the theory of the class of structures of the form $(Q, A; \bar{v}, \rho, \odot)$ where $(Q, A; \bar{v}, \rho) \models T$ and $\odot$ is a compatible A-algebra on Q. Note that this class of structures is elementary.

Lemma 6.20. *The theory of domains and* $T_\odot$ *are existentially bi-intrepretable.*

Proof. We describe an existential bi-interpretation (E, F, η, η'). We let $F(Q, A; \bar{v}, \rho, \odot)$ be A. Let $E(A)$ be $(D(A), A; \bar{u}, \pi_A, \star)$ where $\star$ is the A-algebra on E given by

$$a \star (b_0, \dots, b_m) = (ab_0, \dots, ab_m) \quad \text{for all } a, b_0, \dots, b_m \in A.$$

It is easy to see that E and F are existential interpretations. Note that $\eta_A \coloneqq \mathrm{id}_A$ gives an existentially definable isomorphism between A and $F(E(A))$.

Suppose $P = (Q, A; \bar{v}, \rho, \odot)$ is a model of $T_\odot$. Then $F(P)$ is A and $E(F(P))$ is $(D(A), A; \bar{u}, \rho, \star)$. Let $\vartheta\colon D(A) \to Q$ be defined as in Lemma 6.18. Then $(\vartheta, \mathrm{id}_A)$ is a P-definable isomorphism between $E(F(P))$ and P. We let η'_P be the inverse of $(\vartheta, \mathrm{id}_A)$. ∎

Let $T_{\odot_1}$ be the theory of $(Q, A; \bar{v}, \rho, \odot_1)$ where $(Q, A; \bar{v}, \rho) \models T$ and $\odot_1$ is a compatible A-algebra on Q, likewise for $T_{\odot_2}$. We let $T_\cup = T_{\odot_1} \cup T_{\odot_2}$.

Theorem 6.21. *There is an existential bi-interpretation (E, F, η, η') between the theory of D-rings and $T_\cup$.*

Proof. (Sketch) Suppose $P = (Q, A; \bar{v}, \rho, \odot_1, \odot_2) \models T_\cup$. We describe $F(P)$. Let ϑ_1 and ϑ_2 be the isomorphisms $(D(A); \bar{u}, \pi_A) \to (Q; \bar{v}, \rho)$ associated to $\odot_1$ and $\odot_2$. Note ϑ_1 and ϑ_2 are P-definable. Let $e : A \to Q$ be given by $e(a) = a \odot_1 1_Q$. Then e is a section of ρ. Lemma 6.17 shows that, for all $a \in A$, $e(a)$ is of the form $(a \odot_2 v_0) + (b_1 \odot_2 v_1) + \ldots + (b_m \odot_2 v_m)$ for some $b_1, \ldots, b_m \in A$. Let $\partial_0, \ldots, \partial_m$ be the unique functions $A \to A$ satisfying

$$e(a) = \sum_{i=0}^{m} \partial_i(a) \odot_2 v_i \quad \text{for all } a \in A.$$

Equivalently

$$\vartheta_2(\partial_0(a), \ldots, \partial_m(a)) = e(a) \quad \text{for all} \quad a \in A.$$

Note δ_0 is the identity. As e is a section of ρ Lemma 6.19 shows that $(A; \bar{\partial})$ is a D-ring. Note that $\bar{\partial}$ is P-definable. We let $(A; \bar{\partial})$ be $F(P)$.

Now suppose $(A; \bar{\partial})$ is a D-ring. We describe $E(A; \bar{\partial})$. We define two compatible A-algebras $\star_1, \star_2$ on $D(A)$. We declare

$$a \star_2 (b_0, \ldots, b_{m+1}) = (ab_0, \ldots, ab_m) \text{ for all } a, b_0, \ldots, b_m \in A.$$

Let $\tau \colon A \to D(A)$ be given by

$$\tau(a) = (\partial_0(a), \ldots, \partial_m(a)) = \sum_{i=0}^{m} \partial_0(a) \star_2 u_i.$$

Then τ is a ring homomorphism and a section of π_A. Let $\star_1$ be the A-algebra associated to τ, so that $a \star_1 b = \tau(a)b$ for all $a \in A, b \in D(A)$. Equivalently, we have

$$a \star_1 (b_0, \ldots, b_m) = \left(\sum_{i,j} c_{ij0} \partial_i(a) b_j, \ldots, \sum_{i,j} c_{ijm} \partial_i(a) b_j \right)$$

We declare $E(A; \bar{\partial})$ to be $(D(A), A; \bar{u}, \rho, \star_1, \star_2)$. Then id_A gives an isomorphism between $F(E(A; \bar{\partial}))$ and $(A, \bar{\partial})$ so we let $\eta_{(A, \bar{\partial})}$ be id_A.

Fix $P = (Q, A; \bar{v}, \rho, \odot_1, \odot_2) \models T_\cup$. Let $F(E(P))$ be $(D(A), A; \bar{u}, \rho, \star_1, \star_2)$. Let $\theta : D(A) \to Q$ be given by

$$\theta(a_0, \ldots, a_m) = \sum_{i=0}^{m} a_i \odot_2 v_i.$$

Lemma 6.20 shows that (θ, id_A) gives an isomorphism

$$(D(A), A; \bar{u}, \pi_A, \odot_2) \to (Q, A; \bar{v}, \rho, \star_2).$$

A computation shows that (θ, id_A) also gives an isomorphism $(D(A), A; \bar{u}, \pi_A, \odot_1, \odot_2) \to (Q, A; \bar{v}, \rho, \star_1, \star_2)$. Thus the inverse of (θ, id_A) is an isomorphism between P and $E(F(P))$. So we declare η'_P to be the inverse of (θ, id_A). □

6.5. **Algebraically closed fields with several independent valuations.** A valuation v on a field K is trivial if the v-topology on K is discrete, equivalently if every element of K lies in the valuation ring of v. In this section all valuations are non-trivial. It is well known that the theory ACVF of algebraically closed valued fields admits quantifier elimination and is thus model complete, see for example [HHM06, Theorem 2.1.1]. In this subsection dim is acl-dimension on ACVF. See [vdD89] for Fact 6.22.

Fact 6.22. Suppose $(K; v) \models$ ACVF. Then model-theoretic algebraic closure in $(K; v)$ agrees with field-theoretic algebraic closure in K (which agrees with model-theoretic algebraic closure in K).

It follows that dim is the elementary rank on ACVF induced by Morley rank on ACF, see Section 5.4 for the definition of induced dimension. The following theorem, a well-known special case of [vdD, 3.1.6] follows from Proposition 5.31 and Fact 6.22.

Proposition 6.23. *Let T be ACF and T_i be the theory of algebraically closed fields with valuation v_i for each $i \in I$. Then $T^*_\cup$ exists.*

We now discuss the pseudo-topological axioms satisfied by $T^*_\cup$. The collection of irreducible varieties forms a pseudo-cell collecion for ACF.

Proposition 6.24. *Suppose $(K; v) \models$ ACVF, $V \subseteq K^m$ is an irreducible subvariety, and $X \subseteq V$ is $(K; v)$-definable. Then the following are equivalent:*

(1) $\dim X = \dim V$,
(2) X is Zariski dense (equivalently pseudo-dense) in V,
(3) X has non-empty interior in the v-topology on V.

Proof. Fact 6.22 together with Lemma 5.19 shows that (1) and (2) are equivalent. The proof of [vdD89, Proposition 2.18] shows that (2) implies (3). Every Zariski closed set is v-closed, it follows that any subset of V which is not Zariski dense in V has empty interior in the v-topology on V. □

Suppose K is an algebraically closed field and $\{v_i\}_{i\in I}$ is a family of valuations on K. Proposition 6.24 now yields:

Proposition 6.25. *The following are equivalent:*

(1) $(K; \{v_i\}_{i\in I}) \models T^*_\cup$,
(2) whenever $V \subseteq K^m$ is Zariski closed and irreducible, $J \subseteq I$ is finite, and U_i is a v_i-open subset of V for $i \in J$ we have $\bigcap_{i\in J} U_i \neq \emptyset$.

Proposition 6.25 is essentially proven in [Joh16, 11.2.1, 11.3]. Johnson also shows that Proposition 6.25(2) holds if and only if the v_i are pairwise independent. Recall that two valuations on a field are *independent* if and only if they induce distinct topologies.

6.6. **p-adic valuations on the integers.** Given a prime p we let $v_p(m)$ be the p-adic valuation of a non-zero integer m. We declare $k \preceq_p l$ if $v_p(k) \leq v_p(l)$. Let $\mathbb{P}$ be the set of primes and T_p be the theory of $(\mathbb{Z}; +, \preceq_p)$. The following was shown in [Ad17]:

Theorem 6.26. $(\mathbb{Z}; +, (\preceq_p)_{p\in\mathbb{P}})$ *is model complete.*

In fact is shown in [Ad17] that $(\mathbb{Z};+,(\preceq_p)_{p\in\mathbb{P}})$ admits quantifier elimination in a natural language. This quantifier elimination is used to construct a set S_p of axioms for each $(\mathbb{Z};+,\preceq_p)$ such that $\bigcup_{p\in\mathbb{P}} S_p$ axiomitizes $(\mathbb{Z};+,(\preceq_p)_{p\in\mathbb{P}})$. This shows that $T_\cup = \bigcup_{p\in\mathbb{P}} T_p$ is already model complete so in this case $T^*_\cup = T_\cup$. We obtain the following.

Corollary 6.27. *$(\mathbb{Z};+,(\preceq_p)_{p\in\mathbb{P}})$ is the interpolative fusion of $\{(\mathbb{Z};+,\preceq_p)\}_{p\in\mathbb{P}}$ over $(\mathbb{Z};+)$.*

This naturally raises the following question.

Question 6.28. When in a union of model complete theories model complete?

The additive group of integers supports a canonical rank, described in [Con18], which we denote by dim. It is well known that $(\mathbb{Z};+)$ is superstable but not $\aleph_0$-stable. As $(\mathbb{Z};+)$ is not $\aleph_0$-stable Proposition 5.23 implies that $(\mathbb{Z};+)$ has an elementary extension which admits a non-approximable expansion. Fix a prime p. One can show that $(\mathbb{Z};+,\prec_p)$ is not approximable over $(\mathbb{Z};+)$ by applying the "quasi-coset" decomposition of $(\mathbb{Z};+)$-definable sets given in [Con18, Theorem 4.10] to show that

$$\{(k,l)\in\mathbb{Z}^2 : k\prec_p l\}$$

does not have a pseudo-closure in $\mathbb{Z}^2$. This presents some technical difficulties so we instead let $(Z;+,\prec_p)$ be an $\aleph_1$-saturated elementary expansion of $(\mathbb{Z};+,\prec_p)$ and show that $(Z;+,\prec_p)$ is not approximable over $(Z,+)$.

Let N be an element of Z such that $k\prec_p N$ for all $k\in\mathbb{Z}$. We show that

$$E := \{z\in Z : N\prec_p z\}$$

does not have a pseudo-closure in Z. We make use of the fact that a $(Z;+)$-definable subset of Z is one-dimensional if and only if it is infinite. As every $(\mathbb{Z};+,\prec_p)$-definable subset of $\mathbb{Z}$ is $(\mathbb{Z};+)$-definable [Ad17] we must pass to an elementary extension to obtain a unary set without a pseudo-closure. Quantifier elimination for $(Z;+,0,1)$ implies that every $(Z;+)$-definable subset of Z is a finite union of sets of the form $(kZ+l)\setminus F$ for $k,l\in\mathbb{Z}$ and finite F. This is also a special case of Conant's quasi-coset decomposition.

Thus, if E has a pseudo-closure then E is pseudo-dense in $kZ+l$ for some $k,l\in\mathbb{Z}$. We fix $k\in\mathbb{Z}, l\in\{0,\dots,k-1\}$ and show E is not psudo-dense in $kZ+l$. As E is a subgroup of $(Z;+)$ and $kZ+l$ is a coset of a subgroup, E and $kZ+l$ are disjoint when $l\neq 0$, so it suffices to treat the case when $l=0$. Then $E\subseteq kZ$. Let $k'=pk$ so $v_p(k')=v_p(k)+1$. Then $E\subseteq k'Z\subseteq kZ$. As $v_p(k'm)\geq v_p(k)+1$ and $v_p(k'm+k)=v_p(k)$ for all $m\in\mathbb{Z}$, $k'Z+k$ is disjoint from $k'Z$. Thus, $k'Z+k$ is a one-dimensional definable subset of kZ which is disjoint from E. Hence E is not pseudo-dense in kZ.

6.7. **Algebraically closed fields with multiplicative circular orders.** We describe a family of structures which was considered in [Tra17] with the goal of finding model-theoretic application of number-theoretic results on character sums over finite fields. We refer to that paper for details. A **circular order** on an abelian group G is a ternary relation $\lhd$ on G which is invariant under the group operation and satisfies the following for all $a,b,c\in G$:

(1) if $\lhd(a,b,c)$, then $\lhd(b,c,a)$;

(2) if $\lhd(a,b,c)$, then $\lhd(c,b,a)$;
(3) if $\lhd(a,b,c)$ and $\lhd(a,c,d)$ then $\lhd(a,b,d)$;
(4) if a,b,c are distinct, then either $\lhd(a,b,c)$ or $\lhd(c,b,a)$.

Fix a prime p and a field $\mathbb{F}$ which is an algebraic closure of the field with p elements. Let $\mathbb{F}^\times$ be the multiplicative group of $\mathbb{F}$ and Σ be a restriction of the graph of addition to $\mathbb{F}^\times$. For the rest of the section, let $\lhd$ range over the circular orders on $\mathbb{F}^\times$. With $\mathbb{T}$ the multiplicative group of complex numbers with norm one, it is an easy observation that any such $\lhd$ can be obtained by pulling back the natural clockwise circular order on $\mathbb{T}$ through an injective group homomorphism $\chi : \mathbb{F}^\times \to \mathbb{T}$. This links the structure $(\mathbb{F}^\times;\Sigma,\lhd)$ to the aforementioned character sums results. The main result of [Tra17], while not stated in this form, is essentially the following.

Theorem 6.29. *For all $\lhd$, $(\mathbb{F}^\times;\Sigma,\lhd)$ is a model of the interpolative fusion $T^*_\cup$ of the theories of $(\mathbb{F}^\times;\Sigma)$ and $(\mathbb{F}^\times;\lhd)$ over the theory of $\mathbb{F}^\times$. Moreover, every model of $T^*_\cup$ is elementarily equivalent to $(\mathbb{F}^\times;\Sigma,\lhd)$ for some $\lhd$.*

References

[Ad17] Eran Alouf and Christian d'Elbée, *A new minimal expansion of the integers*, arXiv:1707.07203 (2017).

[BS01] John T. Baldwin and Saharon Shelah, *Model companions of T_{Aut} for stable T*, Notre Dame J. Formal Logic **42** (2001), no. 3, 129–142 (2003). MR 2010177

[Cas11] Enrique Casanovas, *Simple theories and hyperimaginaries*, Lecture Notes in Logic, vol. 44, Association for Symbolic Logic, Chicago, IL; Cambridge Scientific Publishers, Cambridge, 2011.

[CF04] Enrique Casanovas and Rafel Farré, *Weak forms of elimination of imaginaries*, Mathematical Logic Quarterly **50** (2004), no. 2, 126–140.

[CH99] Zoé Chatzidakis and Ehud Hrushovski, *Model theory of difference fields*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 2997–3071. MR 1652269

[CH03] Gregory Cherlin and Ehud Hrushovski, *Finite structures with few types*, Annals of Mathematics Studies, vol. 152, Princeton University Press, Princeton, NJ, 2003. MR 1961194

[Cha97] Zoé Chatzidakis, *Model theory of finite fields and pseudo-finite fields*, Ann. Pure Appl. Logic **88** (1997), no. 2-3, 95–108, Joint AILA-KGS Model Theory Meeting (Florence, 1995). MR 1600887

[CHL85] G. Cherlin, L. Harrington, and A. H. Lachlan, *$\aleph_0$-categorical, $\aleph_0$-stable structures*, Ann. Pure Appl. Logic **28** (1985), no. 2, 103–135. MR 779159

[CKDL17] Pablo Cubides Kovacsics, Luck Darnière, and Eva Leenknegt, *Topological cell decomposition and dimension theory in P-minimal fields*, J. Symb. Log. **82** (2017), no. 1, 347–358. MR 3631291

[Con18] Gabriel Conant, *There are no intermediate structures between the group of integes and presburger arithmetic*, J. Symb. Log. **83** (2018), no. 1, 187–207. MR 3796282

[CP98] Z. Chatzidakis and A. Pillay, *Generic structures and simple theories*, Ann. Pure Appl. Logic **95** (1998), no. 1-3, 71–92. MR 1650667

[d'E18] Christian d'Elbée, *Generic reduct*, arXiv:1810.11722 (2018).

[DMS13] Alfred Dolich, Chris Miller, and Charles Steinhorn, *Extensions of ordered theories by generic predicates*, J. Symbolic Logic **78** (2013), no. 2, 369–387. MR 3145186

[HHM06] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson, *Definable sets in algebraically closed valued fields: elimination of imaginaries*, J. Reine Angew. Math. **597** (2006), 175–236. MR 2264318

[Hod93] Wilfrid Hodges, *Model theory*, Encyclopedia of mathematics and its applications, vol. 42, Cambridge University Press, 1993.

[HP94] Ehud Hrushovski and Anand Pillay, *Groups definable in local fields and pseudo-finite fields*, Israel J. Math. **85** (1994), no. 1–3, 203–262.

[Joh16] William Johnson, *Fun with fields*, ProQuest LLC, Ann Arbor, MI, 2016, Thesis (Ph.D.)–University of California, Berkeley.

[KP00] Byunghan Kim and A. Pillay, *Around stable forking*, Fundamenta Mathematicae **170** (2000), 107–118.

[KS02] Hirotaka Kikyo and Saharon Shelah, *The strict order property and generic automorphisms*, J. Symbolic Logic **67** (2002), no. 1, 214–216. MR 1889545

[KTW] Alex Kruckman, Minh Chieu Tran, and Erik Walsberg, *Interpolative fusions II*, in preparation.

[Mac97] Angus Macintyre, *Generic automorphisms of fields*, Ann. Pure Appl. Logic **88** (1997), no. 2-3, 165–180, Joint AILA-KGS Model Theory Meeting (Florence, 1995). MR 1600899

[MS14] Rahim Moosa and Thomas Scanlon, *Model theory of fields with free operators in characteristic zero*, J. Math. Log. **14** (2014), no. 2, 1450009, 43. MR 3304121

[PT97] Anand Pillay and Akito Tsuboi, *Amalgamations preserving $\aleph_0$-categoricity*, J. Symb. Log. **62** (1997), no. 4, 1070–1074.

[Sim18a] Pierre Simon, *Linear orders in nip theories*, arXiv:1807.07949 (2018).

[Sim18b] ______, *Nip omega-categorical structures: the rank 1 case*, arXiv:1807.07102 (2018).

[SS12] Saharon Shelah and Pierre Simon, *Adding linear orders*, J. Symbolic Logic **77** (2012), no. 2, 717–725. MR 2963031

[SW15] Pierre Simon and Erik Walsberg, *Tame topology over dp-minimal structures*, Notre dame journal of formal logic (2015), to appear.

[Tra17] Minh Chieu Tran, *Tame structures via multiplicative character sums on varieties over finite fields*, ArXiv e-prints (2017).

[TZ13] Katrin Tent and Martin Ziegler, *On the isometry group of the Urysohn space*, Journal of the London Mathematical Society **87** (2013), 289–303.

[vdD] Lou van den Dries, *Model theory of fields*, PhD Thesis, Utrecht, 1978.

[vdD89] ______, *Dimension of definable sets, algebraic boundedness and Henselian fields*, Ann. Pure Appl. Logic **45** (1989), no. 2, 189–209, Stability in model theory, II (Trento, 1987). MR 1044124

[vdD98] ______, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998. MR 1633348

[Win75] Peter M. Winkler, *Model-completeness and Skolem expansions*, 408–463. Lecture Notes in Math., Vol. 498. MR 0540029

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