FRACTALS AND THE MONADIC SECOND ORDER THEORY OF ONE SUCCESSOR

PHILIPP HIERONYMI AND ERIK WALSBERG

Abstract. Let $X \subseteq \mathbb{R}^n$ be closed and nonempty. If the C^k-smooth points of X are not dense in X for some $k \geq 0$, then $(\mathbb{R}, <, +, 0, X)$ interprets the monadic second order theory of $(\mathbb{N}, +1)$. The same conclusion holds if the Hausdorff dimension of X is strictly greater than the topological dimension of X and X has no affine points. Thus, if X is virtually any fractal subset of $\mathbb{R}^n$, then $(\mathbb{R}, <, +, 0, X)$ interprets the monadic second order theory of $(\mathbb{N}, +1)$.

1. Introduction

This paper is a contribution to a larger research enterprise (see [9, 7, 13, 12, 15, 8]) motivated by the following fundamental question:

What is the logical/model-theoretic complexity generated by fractal objects?

Here we will focus on fractal objects defined in first-order expansion of the ordered real additive group $(\mathbb{R}, <, +, 0)$. Throughout this paper $\mathcal{R}$ is a first-order expansion of $(\mathbb{R}, <, +, 0)$, and "definable" without modification means "$\mathcal{R}$-definable, possibly with parameters from $\mathbb{R}$". The main problem we want to address here is:

If $\mathcal{R}$ defines a fractal object, what can be said about logical complexity of $\mathcal{R}$?

The first result in this direction is [15, Theorem B], which states that whenever $\mathcal{R}$ defines a Cantor set (that is, a nonempty compact subset of $\mathbb{R}$ without interior or isolated points), then $\mathcal{R}$ defines an isomorphic copy of the two-sorted first order structure $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. The latter structure is the standard model of monadic second order theory of $(\mathbb{N}, +1)$. We will use $\mathcal{B}$ to denote this structure. As pointed out in [15], while the theory of $\mathcal{B}$ is decidable by [2], the structure does not enjoy any Shelah-style combinatorial tameness properties, such as NIP or NTP2 (see e.g. [20] for definitions). Thus every structure that defines an isomorphic copy of $\mathcal{B}$, can not satisfy these properties either, and for that reason has to be regarded as complicated or wild in the sense of these combinatorial/model-theoretic tameness notions. In this paper, we extend such results to fractal subsets of $\mathbb{R}^n$.

Let $X \subseteq \mathbb{R}^n$ be nonempty. Given $k \geq 0$, a point p on X is **$\mathbf{C^k}$-smooth** if $U \cap X$ is a C^k-submanifold of $\mathbb{R}^n$ for some nonempty open neighbourhood U of p. A point p on X is **affine** if there is an open neighbourhood U of p such that $U \cap X = U \cap H$ for a hyperplane H. We say that $\mathcal{R}$ is of **field-type** if there is an open interval I, definable functions $\oplus, \otimes : I^2 \to I$, and $0_I, 1_I \in I$ such that $(I, <, \oplus, \otimes, 0_I, 1_I)$ is isomorphic to $(\mathbb{R}, <, +, \cdot, 0, 1)$.

Theorem A. *Let X be a nonempty closed definable subset of $\mathbb{R}^n$. If the C^k-smooth points of X are not dense in X for some $k \geq 0$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. If the affine points of X are not dense in X, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ or is of field-type.*

The o-minimal case of Theorem A is proven in [17], where it is essentially shown that any o-minimal expansion of $(\mathbb{R}, <, +, 0)$ that defines a nowhere locally affine set is of field-type [1]. By combining Theorem A with the main result of [13] we obtain Theorem B:

Theorem B. *Let X be a nonempty closed definable subset of $\mathbb{R}^n$ that does not have any affine points. If the topological dimension of X is strictly less then the Hausdorff dimension of X, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

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[1]Recall that $\mathcal{R}$ is o-minimal if every nonempty definable subset of $\mathbb{R}$ is a finite union of open intervals and singletons, and that an o-minimal structure cannot define an isomorphic copy of $(\mathbb{N}, +1)$ by [22, Remark 2.14].

Theorems A and B show that if $\mathcal{R}$ defines essentially any classical fractal, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. Both theorems are essentially optimal. It is known that $\mathcal{B}$ defines isomorphic copies of expansions of $(\mathbb{R}, <, +, 0)$ that in turn define fractal subsets of $\mathbb{R}^n$. Thus the condition "defines an isomorphic copy of $\mathcal{B}$" can not be replaced by any model-theoretical condition not satisfied by $\mathcal{B}$ itself. We describe an example of such an expansion $\mathcal{R}$.

Fix a natural number $r \geq 2$. Let $V_r(x, u, d)$ be the ternary predicate on $\mathbb{R}$ that holds whenever $u = r^n$ for $n \in \mathbb{N}, n \geq 1$ and there is a base r expansion of x with nth digit d. We let $\mathcal{T}_r$ be $(\mathbb{R}, <, +, 0, V_r)$. It is easy to see that $\mathcal{T}_3$ defines the middle-thirds Cantor set, the Sierpinski triangle, and the Menger carpet. It follows easily from the work in [1] that $\mathcal{B}$ defines an isomorphic copy of each $\mathcal{T}_r$ and that $\mathcal{T}_r$ defines an isomorphic copy of $\mathcal{B}$.

We do not know if Theorem A remains true when "C^k" is replaced with "C^∞". Note that by [19], there is an o-minimal expansion of $(\mathbb{R}, <, +, \cdot, 0, 1)$ that defines a function $f : \mathbb{R} \to \mathbb{R}$ which is not C^∞ on a dense definable open subset of $\mathbb{R}$. However, this function is still C^∞ on a dense open subset of $\mathbb{R}$. These considerations lead to the following question:

Questions 1.1. *Is there an o-minimal expansion of* $(\mathbb{R}, <, +, \cdot, 0, 1)$ *that defines a function* $f : \mathbb{R} \to \mathbb{R}$ *which is not* C^∞ *on a dense open subset of* $\mathbb{R}$*?*

It *may* be possible to adapt [16] to construct such an expansion.

There is no precise definition of a fractal subset of $\mathbb{R}^n$. Most subsets X of $\mathbb{R}^n$ which are said to be fractals satisfy the property that the topological dimension of X is strictly less than the Hausdorff dimension of X. It is therefore natural to explore to what extent metric dimensions coincide with topological dimension (and with each other) on definable sets. We will discuss (but not define) three important and well-known metric dimensions: Hausdorff, packing, and Assouad dimension. We refer to [11, 18] for the definitions of these dimensions and the basic facts we apply. It is well-known that

$$\dim X \leq \dim_{\text{Hausdorff}} X \leq \dim_{\text{Packing}} X \leq \dim_{\text{Assouad}} X$$

for all nonempty subsets X of $\mathbb{R}^n$. Here and below $\dim X$ is the topological dimension of X. Essentially all metric dimensions are bounded below by topological dimension and above by Assouad dimension. We let $\mathbb{R}_{\text{Vec}}$ be the ordered vector space $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$.

Theorem 1.2 ([8, Theorem A])**.** *Suppose* $\mathcal{R}$ *expands* $\mathbb{R}_{\text{Vec}}$ *and* $X \subseteq \mathbb{R}^n$ *is nonempty, closed, and definable. If the topological dimension of* X *is strictly less than the Hausdorff dimension of* X*, then* $\mathcal{R}$ *defines every bounded Borel subset of every* $\mathbb{R}^n$*.*

Observe that whenever $\mathcal{R}$ defines every bounded Borel subset of every $\mathbb{R}^n$, it also defines an isomorphic copy of $\mathcal{B}$. We converse fails as is witnessed by $\mathcal{T}_r$. Thus Theorem A can be seen as an analogue of Theorem 1.2 when $\mathcal{R}$ does not necessarily expand $\mathbb{R}_{\text{Vec}}$. Note that there are compact subsets X of $\mathbb{R}$ with topological dimension zero and positive packing dimension such that $(\mathbb{R}_{\text{Vec}}, X)$ does not define all bounded Borel sets (see [8, Section 7.2]). There are stronger results for expansions of the real field.

Theorem 1.3 ([13, Theorem A])**.** *Suppose* $\mathcal{R}$ *expands* $(\mathbb{R}, <, +, \cdot, 0, 1)$ *and* $X \subseteq \mathbb{R}^n$ *is nonempty, closed, and definable. If the topological dimension of* X *is strictly less than the Assouad dimension of* X*, then* $\mathcal{R}$ *defines every Borel subset of every* $\mathbb{R}^n$*.*

We finish with two open questions.

Questions 1.4. *Let* X *be a nonempty closed definable subset of* $\mathbb{R}^n$*. If the topological dimension of* X *is strictly less than the Hausdorff dimension of* X*, then must* $\mathcal{R}$ *define an isomorphic copy of* $\mathcal{B}$*?*

Observe that Theorem B gives an affirmative answer to Question 1.4 under the additional assumption that X does not have any affine points. We do not even know the answer to the following weaker question.

Questions 1.5. *If* $\mathcal{R}$ *defines an uncountable nowhere dense subset of* $\mathbb{R}$*, must* $\mathcal{R}$ *define an isomorphic copy of* $\mathcal{B}$*? Weaker: if* $\mathcal{R}$ *defines an uncountable nowhere dense subset of* $\mathbb{R}$*, must* $\mathcal{R}$ *have Shelah's independence property?*

The reader may wonder if the C^k points of a definable set always form a definable set. This is indeed true. The proof is non-trivial and not included in the present paper.

2. Conventions, Notation

Throughout m, n are natural numbers, i, j, k, l are integers and $s, t, \delta, \varepsilon$ are real numbers. Throughout "dimension" is topological dimension unless stated otherwise. Let X be a subset of $\mathbb{R}^n$. Then $\dim X$ is the dimension of X, $\mathrm{Cl}(X)$ and $\mathrm{Int}(X)$ are the closure and interior of X, and $\mathrm{Bd}(X) := \mathrm{Cl}(X) \setminus \mathrm{Int}(X)$ is the boundary of X. Given $A \subseteq \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$ we let

$$A_x := \{y \in \mathbb{R}^n : (x, y) \in A\}.$$

We let $\Gamma(f)$ be the graph of a function f and let $f|_Z$ be the restriction of f to a subset Z of its domain. A family $(A_t)_{t>0}$ of sets is **increasing** if $s < t$ implies $A_s \subseteq A_t$ and **decreasing** if $s < t$ implies $A_t \subseteq A_s$.

Throughout $\|,\|$ is the ℓ_∞-norm and an "open ball" is an open ℓ_∞-ball. We use the ℓ_∞-norm (as opposed to the ℓ_2-norm) as it is $(\mathbb{R}, <, +, 0)$-definable. All dimensions of interest are bilipschitz invariants and therefore unaffected by choice of norm.

3. Background

We review definitions and results from the theory of first order expansions of $(\mathbb{R}, <, +, 0)$. An **ω-orderable set** is a definable set that is either finite or admits a definable order of order-type ω. One should think of "ω-orderable sets" as "definably countable sets". A **dense ω-order** is an ω-orderable subset of $\mathbb{R}$ that is dense in some nonempty open interval. We say $\mathcal{R}$ is **type A** if it does not admit a dense ω-order, **type C** if it defines every bounded Borel subset of every $\mathbb{R}^n$, and **type B** if it is neither type A nor type C. It is easy to see that these three classes of structures are mutually exclusive. The first claim of Theorem 3.1 is the main result of [15], the latter claims are proven in [14].

Theorem 3.1. *Let $U \subseteq \mathbb{R}^m$ be a definable open set.*

1. *If $\mathcal{R}$ is not type A (defines a dense ω-orderable set), then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*
2. *If $\mathcal{R}$ is type B, then $\mathcal{R}$ is not of field-type.*
3. *If $\mathcal{R}$ is type A, $k \geq 1$, and $f : U \to \mathbb{R}^n$ is continuous and definable, then there is a dense open subset V of U on which f is C^k.*
4. *If $\mathcal{R}$ is type A and not of field-type, and $f : U \to \mathbb{R}^n$ is definable and continuous, then there is dense open subset V of U on which f is locally affine.*

A subset X of $\mathbb{R}^n$ is $\mathbf{D}_{\boldsymbol{\Sigma}}$ if $X = \bigcup_{s,t>0} X_{s,t}$ for a definable family $(X_{s,t})_{s,t>0}$ of compact subsets of $\mathbb{R}^n$ such that $X_{s,t} \subseteq X_{s,t'}$ when $t \leq t'$ and $X_{s',t} \subseteq X_{s,t}$ when $s \leq s'$. We say that such a family **witnesses** that X is D_Σ. Note that a D_Σ set is definable and that every D_Σ set is F_σ. This is not obvious, but one should think of D_Σ sets as "definably F_σ sets".

Proposition 3.2 ([3, 1.10])**.** *Open and closed definable sets are D_Σ, a finite union or finite intersection of D_Σ sets is D_Σ, and the image of a D_Σ-set under a continuous definable function is D_Σ.*

Proposition 3.3 below is known as the strong Baire category theorem, or **SBCT**.

Proposition 3.3 ([8, Theorem 4.1])**.** *Suppose $\mathcal{R}$ is type A. Let X be a D_Σ subset of $\mathbb{R}^n$ witnessed by the definable family $(X_{s,t})_{s,t>0}$. Then X either has interior or is nowhere dense. If X has interior then $X_{s,t}$ has interior for some $s, t > 0$. Furthermore, if $(X_t)_{t>0}$ is an increasing family of D_Σ sets and $\bigcup_{t>0} X_t$ has interior then X_t has interior for some $t > 0$.*

Note that the latter two claims follow by applying the Baire category theorem to the first claim. Corollary 3.4 below follows from SBCT and the fact that the closure and interior of a D_Σ set are D_Σ.

Corollary 3.4. *Let $X \subseteq \mathbb{R}^n$ be D_Σ. Then $\mathrm{Bd}(X)$ is nowhere dense.*

We refer to Proposition 3.5 below as **$\mathbf{D}_{\boldsymbol{\Sigma}}$-selection**.

Proposition 3.5 ([8, Proposition 5.5])**.** *Suppose $\mathcal{R}$ is type A. Let $X \subseteq \mathbb{R}^{m+n}$ be D_Σ, and $U \subseteq \mathbb{R}^m$ be a nonempty open set contained in the coordinate projection of X onto $\mathbb{R}^m$. Then there is a nonempty definable open $V \subseteq U$ and a continuous definable $f : V \to \mathbb{R}^n$ such that $\Gamma(f) \subseteq X$.*

Proposition 3.6 and Proposition 3.7 are special cases of more general results on additivity of dimension [8, Theorem E].

Proposition 3.6. *Suppose $\mathcal{R}$ is type A. If $A \subseteq \mathbb{R}^n$ is D_Σ of dimension $1 \leq d \leq n-1$ then there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and a nonempty open $U \subseteq \mathbb{R}^d$ contained in $\pi(A)$ such that A_x is zero-dimensional for all $x \in U$.*

Proposition 3.7 below shows that type A expansions cannot define space-filling curves [8, Theorem E].

Proposition 3.7. *Suppose $\mathcal{R}$ is type A. Let $X \subseteq \mathbb{R}^n$ be* D_Σ. *Then* $\dim f(X) \leq \dim X$ *for any continuous definable* $f : X \to \mathbb{R}^m$.

For our purposes a **Cantor subset** is a nonempty compact nowhere dense subset of $\mathbb{R}$ without isolated points. If $X \subseteq \mathbb{R}$ then $p \in X$ is C^k-smooth (for any $k \geq 0$) if and only if p is either isolated in X or lies in the interior of X. Therefore Proposition 3.8 below yields Theorem A for definable subsets of $\mathbb{R}$.

Proposition 3.8 ([15, Theorem B])**.** *If $\mathcal{R}$ defines a Cantor subset of $\mathbb{R}$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

Finally, we following generalization of Theorem 1.3 holds.

Proposition 3.9 ([14, Theorem 6.2])**.** *Suppose $\mathcal{R}$ is type A and of field-type. Let $X \subseteq \mathbb{R}^n$ be nonempty,* D_Σ*, and bounded. Then the Assouad dimension of X agrees with the topological dimension of X.*

4. Hausdorff continuity of definable families

Throughout this section $\mathcal{R}$ is assumed to be type A and U is a fixed nonempty definable open subset of $\mathbb{R}^m$. We recall some useful metric notions. Suppose $f : (X, d_X) \to (Y, d_Y)$ is a function between metric spaces. The **oscillation** of f at $x \in X$ is the supremum of all $\delta \geq 0$ such that for every $\varepsilon > 0$ there are $z, z' \in X$ such that $d_X(x, z), d_X(x, z') < \varepsilon$ and $d_Y(f(z), f(z')) > \delta$. Recall that f is continuous at x if and only if the oscillation of f at x is zero and that the set of $x \in X$ at which the oscillation of f is $\geq \varepsilon$ is closed for every $\varepsilon > 0$.

The **Hausdorff distance** $d_{\mathcal{H}}(A, B)$ between bounded subsets A and B of $\mathbb{R}^n$ is the infimum of $\delta > 0$ such that for every $a \in A$ there is a $b \in B$ such that $\|a - b\| < \delta$ and for every $b \in B$ there is an $a \in A$ such that $\|a - b\| < \delta$. The Hausdorff distance between a bounded subset of $\mathbb{R}^n$ and its closure is zero. The Hausdorff distance restricts to a separable complete metric on the collection $\mathcal{C}$ of all compact subsets of $\mathbb{R}^m$. Lemma 4.1 below follows directly from the definition of $d_{\mathcal{H}}$.

Lemma 4.1. *Let W be a bounded open subset of $\mathbb{R}^n$, $\mathcal{D}$ is a collection of open balls of diameter $\leq \varepsilon$ covering W, and A and A' be subsets of W. If*
$$\{B \in \mathcal{D} : B \cap A \neq \emptyset\} = \{B \in \mathcal{D} : B \cap A' \neq \emptyset\},$$
then $d_{\mathcal{H}}(A, A') \leq \varepsilon$.

Given $Z \subseteq \mathbb{R}^m$ and a family $\mathcal{A} = (A_x)_{x \in Z}$ of subsets of $\mathbb{R}^n$, we let $M_{\mathcal{A}} : Z \to \mathcal{C}$ be given by $M_{\mathcal{A}}(x) = \mathrm{Cl}(A_x)$. We say that $\mathcal{A}$ is **HD-continuous** if $M_{\mathcal{A}}$ is continuous. We let $\mathcal{O}_\varepsilon(\mathcal{A})$ be the set of points in Z at which $M_{\mathcal{A}}$ has oscillation at least $\varepsilon > 0$ and let $\mathcal{O}(\mathcal{A})$ be the set of points at which $M_{\mathcal{A}}$ has positive oscillation. The complement of $\mathcal{O}(\mathcal{A})$ is the set of points at which $M_{\mathcal{A}}$ is continuous and each $\mathcal{O}_\varepsilon(\mathcal{A})$ is closed in Z.

We say that $A \subseteq \mathbb{R}^{m+n}$ is **vertically bounded** if there is an open ball W in $\mathbb{R}^n$ such that $A_x \subseteq W$ for all $x \in \mathbb{R}^m$.

Proposition 4.2. *Suppose $A \subseteq U \times \mathbb{R}^n$ is* D_Σ *and vertically bounded. Let $\mathcal{A}$ be the definable family $(A_x)_{x \in U}$ of subsets of $\mathbb{R}^n$. Then there is a dense definable open subset U' of U such that $(A_x)_{x \in U'}$ is HD-continuous.*

In the proof of Proposition 4.2 below π is the coordinate projection $U \times \mathbb{R}^m \to U$.

Proof. Note that it suffices to show that every point in U has a neighbourhood V such that the proposition holds for the restricted family $(A_x)_{x \in V}$. We may therefore assume that U is an open ball and in particular is connected. We show that $\mathcal{O}(\mathcal{A})$ is nowhere dense and take U' to be the interior of the complement of $\mathcal{O}(\mathcal{A})$ in U. Since $(\mathcal{O}_\varepsilon(\mathcal{A}))_{\varepsilon > 0}$ witnesses that $\mathcal{O}(\mathcal{A})$ is D_Σ, it suffices to show $\mathcal{O}_\varepsilon(\mathcal{A})$ is nowhere dense for all $\varepsilon > 0$ and apply SBCT.

Fix $\varepsilon > 0$. Let W be an open ball in $\mathbb{R}^n$ such that $A_x \subseteq W$ for all $x \in U$. Let $\mathcal{D}$ be a finite collection of closed balls of diameter $\leq \varepsilon$ covering W. Set
$$E_B := \pi([\mathbb{R}^n \times B] \cap A) \quad \text{for each} \quad B \in \mathcal{D}.$$
That is, E_B is the set of $x \in U$ such that A_x intersects B. Proposition 3.2 shows that each E_B is D_Σ. Corollary 3.4 shows that $\mathrm{Bd}(E_B)$ is nowhere dense in U for each $B \in \mathcal{D}$. Recall that the boundary of a subset of a topological space is always closed. Therefore
$$V := \bigcap_{B \in \mathcal{D}} U \setminus \mathrm{Bd}(E_B)$$

is dense and open in U. We show that V is a subset of $\mathcal{O}_\varepsilon(\mathcal{A})$. We fix $p \in V$ and show that the oscillation of $M_{\mathcal{A}}$ at p is $\leq \epsilon$. Let R be an open ball with center p contained in V. As R and U are both connected, and $R \cap \mathrm{Bd}(E_B) = \emptyset$ for all $B \in \mathcal{D}$, R is either contained in, or disjoint from, each E_B. Fix $q \in R$. Then $q \in E_B$ if and only if $p \in E_B$ for all $B \in \mathcal{D}$. That is, A_q intersects any $B \in \mathcal{D}$ if and only if A_p intersects B. Lemma 4.1 now shows that $d_{\mathcal{H}}(A_p, A_q) \leq \varepsilon$.

□

We say that a point p on a subset X of $\mathbb{R}^n$ is **ε-isolated** if $\|p - q\| \geq \varepsilon$ for all points $q \neq p$ on X. We leave the verification of the following lemma, an exercise of metric geometry, to the reader.

Lemma 4.3. *Fix $\varepsilon > 0$. Let A be a vertically bounded subset of $U \times \mathbb{R}^n$ such that $(A_x)_{x\in U}$ is HD-continuous. Then the set of $(x, y) \in A$ such that y is ε-isolated in A_x is closed in A.*

Corollary 4.4 follows from Lemma 4.3 as a definable set that is closed in a D_Σ-set, is D_Σ.

Corollary 4.4. *Let A be a vertically bounded D_Σ subset of $U \times \mathbb{R}^n$ such that $(A_x)_{x\in U}$ is HD-continuous. Then the set of $(x, y) \in A$ such that y is ε-isolated in A_x is D_Σ.*

5. C^k-smooth points on D_Σ-sets

We prove Theorem A and Theorem B in this section. Recall that if $\mathcal{R}$ is not type A, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ by Theorem 3.1(1). If $\mathcal{R}$ defines a Cantor subset of $\mathbb{R}$ then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ by [15, Theorem B]. Thus it suffices to show that Theorem A and Theorem B hold under the assumption that $\mathcal{R}$ is type A and does not define a Cantor subset of $\mathbb{R}$. We do so in Proposition 5.4 and Proposition 5.6 below. We suppose throughout this section that $\mathcal{R}$ is type A.

Lemma 5.1 is an elementary fact of real analysis, we leave the details to the reader. (Take V such that $\|f(x) - f(y)\| < \frac{\varepsilon}{2}$ for all $x, y \in V$).

Lemma 5.1. *Let $A \subseteq \mathbb{R}^{m+n}$, $U \subseteq \mathbb{R}^m$ be nonempty open, $\varepsilon > 0$, and $f : U \to \mathbb{R}^n$ be continuous such that $f(x)$ is an ε-isolated element of A_x for all $x \in U$. Then there are nonempty open $V \subseteq U$ and $W \subseteq \mathbb{R}^n$ such that $A \cap [V \times W] = \Gamma(f|_V)$.*

Lemma 5.2. *Let $k \geq 0$, $A \subseteq \mathbb{R}^{m+n}$ be D_Σ, and U be a nonempty definable open set contained in the coordinate projection of A onto $\mathbb{R}^m$ such that the isolated points of A_x are dense in A_x for all $x \in U$. Then there are nonempty definable open $V \subseteq U$ and $W \subseteq \mathbb{R}^n$ and a definable C^k-function $f : V \to W$ such that*

$$A \cap [V \times W] = \Gamma(f).$$

If $\mathcal{R}$ is not of field-type, then we may take f to be affine.

In the proof that follows $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the coordinate projection onto the first m coordinates.

Proof. We first reduce to the case when A is vertically bounded. Given $r > 0$ let

$$A^r = \{(x, y) \in A : \|y\| < r\}$$

so that $\bigcup_{r>0} A^r = A$. Then $(\pi(A^r))_{r>0}$ is an increasing definable family of D_Σ-sets and U is contained in $\bigcup_{r>0} \pi(A^r)$. By SBCT $\pi(A^t)$ has interior for some $t > 0$. After replacing U with $\mathrm{Int}(A^t)$ and A with A^t if necessary we suppose A is vertically bounded. After applying Lemma 4.2 and replacing U with a smaller nonempty definable open set we suppose $(A_x)_{x\in U}$ is HD-continuous. For each $\varepsilon > 0$ let $S_\varepsilon \subseteq A$ be the set of (x, y) such that y is ε-isolated in A_x. Corollary 4.4 shows each S_ε is D_Σ. Then $U \subseteq \bigcup_{\varepsilon>0} \pi(S_\varepsilon)$ as A_x has an isolated point for each $x \in U$. As $(\pi(S_\epsilon))_{\epsilon>0}$ is an increasing family of D_Σ sets SCBT gives a $\delta > 0$ such that $\pi(S_\delta)$ has interior in U. After replacing U with a smaller nonempty definable open set if necessary we suppose U is contained in $\pi(S_\delta)$. Applying D_Σ-selection we obtain a nonempty definable $V \subseteq U$ and a continuous definable $f : V \to \mathbb{R}^n$ such that $(x, f(x)) \in S_\delta$ for all $x \in V$. Thus $f(x)$ is δ-isolated in A_x for all $x \in V$. After applying Theorem 3.1(3) and shrinking V if necessary we suppose f is C^k. Now apply Lemma 5.1.

If $\mathcal{R}$ is not of field-type, then after applying Theorem 3.1(4) and shrinking V if necessary we may suppose f is affine. □

Lemma 5.3. *Suppose $\mathcal{R}$ does not define a Cantor subset of $\mathbb{R}$. Let $A \subseteq \mathbb{R}^n$ be D_Σ. If $\dim A = 0$, then the isolated points of A are dense in A.*

It is easy to see that $\mathcal{R}$ defines a Cantor subset of $\mathbb{R}$ if and only if it defines a nowhere dense subset of $\mathbb{R}$ without isolated points (take closures).

Proof. We apply induction on n. Suppose $n = 1$. Let U be an open set that intersects A. Then $A \cap U$ contains an isolated point. Thus the isolated points of A are dense in A. Now suppose $n > 1$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection onto the first n coordinates. Proposition 3.7 shows that $\dim \pi(A) = 0$. Induction and the fact that $\pi(A)$ is D_Σ implies that $\pi(A)$ contains an isolated point x. Then A_x is a definable zero-dimensional subset of $\mathbb{R}$ and thus contains an isolated point t. It is easy to see that (x, t) is isolated in A. □

Proposition 5.4. *Suppose $\mathcal{R}$ does not define a Cantor subset of $\mathbb{R}$. Let $A \subseteq \mathbb{R}^n$ be D_Σ and nonempty. Then the C^k-smooth points of A are dense in A. If $\mathcal{R}$ is in addition not of field-type, then the affine points of A are dense in A.*

In the proof below we apply the fact [4, Theorem 1.8.10] that an arbitrary subset of $\mathbb{R}^n$ is n-dimensional if and only if it has nonempty interior.

Proof. Let $U \subseteq \mathbb{R}^n$ be a definable open set that intersects A. We show U contains a C^k-smooth point of A. Let d be the dimension of $U \cap A$. If $d = 0$, apply Lemma 5.3. If $d = n$, then A has interior in U, and every interior point of A is C^k-smooth. Suppose $1 \leq d \leq n - 1$. Applying Proposition 3.6 there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and a nonempty definable open $V \subseteq \mathbb{R}^d$ such that $V \subseteq \pi(A)$ and $\dim A_x = 0$ for all $x \in V$. As $\mathcal{R}$ does not define a Cantor subset, the isolated points of A_x are dense in A_x for all $x \in V$. Apply Lemma 5.2. □

We now prove Theorem B. Theorem A, together with Proposition 3.9, and Theorem 3.1(1), yields Proposition 5.5.

Proposition 5.5. *Let $X \subseteq \mathbb{R}^n$ be nonempty, D_Σ, and bounded such that the affine points of X are not dense in X. If the topological dimension of X is strictly less than the Assouad dimension of X, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

Proposition 5.5 does not generalize to unbounded D_Σ sets. Marker and Steinhorn showed that $(\mathbb{R}, <, +, 0, \sin)$ is locally o-minimal, and the proof described in [21, Theorem 2.7] shows that this expansion does not have the independence property. Then the $(\mathbb{R}, <, +, 0, \sin)$-definable set

$$\{(x, t + \sin(x)) : t \in \pi\mathbb{Z}, x \in \mathbb{R}\}$$

has Assouad dimension two, topological dimension one, and no affine points. The result below implies Theorem B.

Proposition 5.6. *Let $X \subseteq \mathbb{R}^n$ be D_Σ such that X has no affine points. If the topological dimension of X is strictly less than the packing dimension of X, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

Proof. We let $\dim_{\mathrm{Pack}} X$ be the packing dimension of a subset X of $\mathbb{R}^n$ in the proof below. It follows directly from the definition of packing dimension that whenever $Y \subseteq \mathbb{R}^n$ is Borel, and $(Y_m)_{m \in \mathbb{N}}$ is a collection of Borel subsets of Y covering Y, then $\dim_{\mathrm{Pack}} Y = \sup_m \dim_{\mathrm{Pack}} Y_m$. The same statement holds for topological dimension provided each Y_m is F_σ by [4, Corollary 1.5.4].

Given $\bar{m} \in \mathbb{Z}^n$ we let $X_{\bar{m}}$ be $([0,1]^n + \bar{m}) \cap X$. As each $X_{\bar{m}}$ is F_σ we have $\dim X = \sup_{\bar{m} \in \mathbb{Z}^n} \dim X_{\bar{m}}$. Thus if $\dim X_{\bar{m}} = \dim_{\mathrm{Pack}} X_{\bar{m}}$ for all $\bar{m} \in \mathbb{Z}^n$ we have $\dim X = \dim_{\mathrm{Pack}} X$. If $\dim X_{\bar{m}} < \dim_{\mathrm{Pack}} X_{\bar{m}}$ for some $\bar{m} \in \mathbb{Z}^n$, then, as each $X_{\bar{m}}$ has no affine points, Proposition 5.5 shows that $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. □

6. A type A expansion without dimension coincidence

Whenever $\mathcal{R}$ is type B, we know that $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ by Theorem 3.1. In this section we show that in Theorem B the statement "defines an isomorphic copy of $\mathcal{B}$" can not be replaced by the stronger statement "is type B". We do so by giving an example of a type A expansion that defines a compact zero-dimensional subset of $\mathbb{R}$ with positive Hausdorff dimension. Our construction is an application of a theorem of Friedman and Miller [10].

Proposition 6.1. *Let $\mathcal{M}$ be an o-minimal expansion of $(\mathbb{R}, <, +, 0)$ and let $E \subseteq \mathbb{R}$ be closed and nowhere dense. Then the following are equivalent:*

1. *$(\mathcal{M}, E)$ is type A,*
2. *every $(\mathcal{M}, E)$-definable subset of $\mathbb{R}$ has interior or is nowhere dense,*

(3) $f(E^n)$ is nowhere dense for every $\mathcal{M}$-definable $f : \mathbb{R}^n \to \mathbb{R}$.

Proof. A dense ω-orderable set is dense and co-dense in some nonempty open interval, hence (2) implies (1). Friedman and Miller [10] showed (3) implies (2). Proposition 3.7 shows that (1) implies (3). □

Recall $(\mathbb{R}, <, +, 0)$ is o-minimal. The following lemma can be deduced easily from the semi-linear cell decomposition (see [22, Corollary 1.7.8]) for $(\mathbb{R}, <, +, 0)$-definable sets. We leave the details to the reader.

Lemma 6.2. *Let $E \subseteq \mathbb{R}$. The following are equivalent:*

(1) $f(E^n)$ is nowhere dense for every $(\mathbb{R}, <, +, 0)$-definable $f : \mathbb{R}^n \to \mathbb{R}$,
(2) $T(E^n)$ is nowhere dense for every $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$.

We now characterize compact zero-dimensional subsets E of $\mathbb{R}$ such that $(\mathbb{R}, <, +, 0, E)$ is type A.

Theorem 6.3. *Let $E \subseteq \mathbb{R}$ be compact and nowhere dense. Then the following are equivalent:*

(1) $(\mathbb{R}, <, +, 0, E)$ is type A,
(2) every $(\mathbb{R}, <, +, 0, E)$-definable subset of $\mathbb{R}$ has interior or is nowhere dense,
(3) $T(E^n)$ is nowhere dense for every $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$,
(4) the subgroup of $(\mathbb{R}, +, 0)$ generated by E is not equal to $\mathbb{R}$.

Proof. The equivalence of (1), (2), and (3) follows by Fact 6.1 and Lemma 6.2. We show that (3) and (4) are equivalent. Let

$$E_{\bar{m}} := \{m_1 e_1 + \ldots + m_n e_n : e_1, \ldots, e_n \in E\}$$

for every $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Then $\bigcup_{n>0} \bigcup_{\bar{m} \in \mathbb{Z}^n} E_{\bar{m}}$ is the subgroup of $(\mathbb{R}, +, 0)$ generated by E. If $T(E^n)$ is nowhere dense for all $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$ then $E_{\bar{m}}$ is nowhere dense and $\bigcup_{n>0} \bigcup_{\bar{m} \in \mathbb{Z}^n} E_{\bar{m}}$ is meager.

We show that (4) implies (3) by contrapositive. Fix $\bar{q} = (q_1, \ldots, q_n) \in \mathbb{Q}^n$, and let

$$T(\bar{x}) = q_1 x_1 + \ldots + q_n x_n \quad \text{for all } \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

and suppose that $T(E^n)$ is somewhere dense. Then $T(E^n)$ is compact as E^n is compact, so $T(E^n)$ has interior. Let $m \in \mathbb{Z}$ be such that $mq_i \in \mathbb{Z}$ for all $1 \leq i \leq n$ and let $m\bar{q} = (mq_1, \ldots, mq_n)$. Then

$$E_{m\bar{q}} = \{mq_1 e_1 + \ldots + mq_n e_n : e_1, \ldots, e_n \in E\} = mT(E^n)$$

has interior. So the subgroup of $(\mathbb{R}, +, 0)$ generated by E has interior and therefore equals $\mathbb{R}$. □

Note that the subgroup of $(\mathbb{R}, +, 0)$ generated by E is F_σ and therefore Borel. There are examples, first due to Erdös and Volkmann [5], of proper Borel subgroups of $(\mathbb{R}, +, 0)$ with positive Hausdorff dimension. If G is a proper Borel subgroup of $(\mathbb{R}, +, 0)$ with positive Hausdorff dimension, then inner regularity of Hausdorff measure [18, Corollary 4.5] yields a compact subset of G with positive Hausdorff dimension. Such subsets necessarily have empty interior and are therefore zero-dimensional. See [6, Example 12.4] for specific examples of compact subsets of $\mathbb{R}$ which generate proper subgroups of $(\mathbb{R}, +, 0)$.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

Email address: phierony@illinois.edu
URL: http://www.math.illinois.edu/~phierony

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

Email address: erikw@illinois.edu
URL: http://www.math.illinois.edu/~erikw