Consider $f : U \to Y$, $U \subseteq X \times \Lambda$ open, $X, \Lambda, Y$ Banach spaces and assume

$$f(x_0, \lambda_0) = 0, \quad (x_0, \lambda_0) \in U$$

Study the solutions of

$$f(x, \lambda) = 0 \quad \text{in } U$$

Assume $f$ is $C^1$ on $U$ and $D_\lambda f(x_0, \lambda_0)$ is Fredholm.

(a) ker $D_\lambda f(x_0, \lambda_0) = X_1$ is finite dimensional.

(b) range $D_\lambda f(x_0, \lambda_0) = Y_1$ is a closed subspace of $Y$ with finite codimension.

The procedure reduces the infinite dimensional equation (1) to a finite dimensional equation:

Consider the decompositions

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2$$

with associated continuous projections $P : X \to X_2$ and $Q : Y \to Y_1$

(1) is equivalent to:

$$\begin{align*}
Qf(x_1 + x_2, \lambda) &= 0 \quad (2.1) \\
(Id - Q)f(x_1 + x_2, \lambda) &= 0 \quad (2.2)
\end{align*}$$

Let $x_0 = x_1^0 + x_2^0$, $x_1^0 \in X_1$, $x_2^0 \in X_2$ then

exists $n_1, n_2, n_\infty$ such that $(x_1 + x_2, \lambda) \in U$ whenever

$$(x_2, x_1, \lambda) \notin B_{n_2}(x_2^0) \times B_{n_1}(x_1^0) \times B_{n_\infty}(\lambda_0) \subseteq X_2 \times X_1 \times \Lambda$$
Let \( F : B_{\mathbb{R}^2}(x_0) \times B_{\mathbb{R}^2}(x_0) \times \mathbb{R} \to Y \) given by
\[
F(x_2, x_1, n) = Q f(x_1 + x_2, n)
\]
Then
\[
F(x_2^0, x_1^0, n_0) = Q f(x_0, n_0) = Q 0 = 0
\]
\[
D_{x_2} F(x_2^0, x_1^0, n_0) = Q D_{x_2} f(x_0, n_0) \cdot i_{x_1}^0
\]
where \( i_{x_1}^0 : X_2 \to X \) \( i_{x_1}^0(x_2) = x_2 \)

We have
\[
\ker D_{x_2} F(x_2^0, x_1^0, n_0) = \{ 0 \}
\]
because 
\[
\ker i_{x_1} = \{ 0 \} \quad \text{ Range } i_{x_1} = X_2 \quad \text{and} \quad X_1 \cap \ker D_{x_1} F = \{ 0 \}
\]
and 
\[
\text{range } D_{x_1} f = Y, \quad \text{on which } Q 0 \text{ is the identity map.}
\]
Moreover 
\[
\text{range } D_{x_2} F(x_2^0, x_1^0, n_0) = Y
\]
\[
D_{x_2} F \in B(X_2, Y)
\]
Hence \( D_{x_2} F(x_2^0, x_1^0, n_0) : X_2 \to Y \) is linear, \( C^1 \) continuous and injective. By the open mapping theorem it is an isomorphism.

Since \( F \) is \( C^1 \) on \( B_{\mathbb{R}^2}(x_0) \times B_{\mathbb{R}^2}(x_0) \times \mathbb{R} \) the Implicit Function Theorem, applied to (2.1) gives
\[
0 < \rho_2 \leq n_2, \quad 0 < \rho_1 \leq n_1, \quad 0 < n_1 < n_2 \text{ and a } C^1 \text{ map}
\]
\[
x_2 : B_{\mathbb{R}_2}(\rho_2)(x_0) \times B_{\mathbb{R}_2}(\rho_2)(n_0) \to B_{\mathbb{R}_2}(\rho_2)(x_0)
\]
Such that
\[(x_2(x_1, x_2), x_1, x_2)\] are the only subs of (2.1) in \(V_{\mathcal{O}} \times U_{\mathcal{O}} \times U_{\mathcal{O}}\) neighborhood of \((x_0, x_0)\)

Plugging in (2.2) we get:
\[(\text{Id}-Q)\mathcal{L}((x_1 + x_2(x_1, x_2)), x_2) = 0\]

The left hand side is \(C^1\) in \(x, x_2, \lambda\) and defined on a subset of \(X_1 \times X_2\) with values in \(Y_2\).

\(X_1\) and \(Y_2\) are both finite dimensional.

Note that: For \(G(x, \lambda) = (\text{Id}-Q)\mathcal{L}(x + x_2(x_1, x_2), x_2)\)
we have:
\[G(x_0, \lambda_0) = (\text{Id}-Q)\mathcal{L}(x_0, \lambda_0) = 0\] and
\[\partial_{x_1} G(x_0, \lambda_0) = (\text{Id}-Q)\mathcal{L}_{x_1}(x_0, \lambda_0) \circ (\text{Id} + \partial_{x_2} x_2) = 0\]

Because range of \(\mathcal{L}_{x_1}(x_0, \lambda_0) = Y_1 = \ker(\text{Id}-Q)\)

So if one wants to use again implicit function theorem for (2.2) then one should look for
\[\lambda = \lambda(x_1);\]
\[\partial_{x_1} G(x_0, \lambda_0) = (\text{Id}-Q)\mathcal{L}_{x_1}(x_0, \lambda_0) \circ (\text{Id} + \partial_{x_2} x_2)\]
\[+ (\text{Id}-Q)\mathcal{L}_{x_2}(x_0, \lambda_0),\]

Conclusion: The Lyapunov-Schmidt relation makes a one to one correspondence between the solutions of \(f(x, \lambda) = 0, x \in X_1, \lambda \in \Lambda\) and the solutions of the \(Y_1\) equations in \(x_2, \lambda\). Provided \(f\) is locally \(C^1\) and \(\partial_{x_1} f(x_0, \lambda_0)\) is Fredholm.
Suppose Schrödinger reduction with symmetries

We say that a compact group $\Gamma$ acts on a Banach space $X$ if there exists a continuous homomorphism

$$\Phi : (\Gamma, \cdot) \rightarrow (GL(B(X, X)), 0)$$

where $GL(B(X, X)) = \{ L : X \rightarrow X \mid L \text{ linear, bounded and with bounded inverse} \}$

Notation: $y \cdot x = \Phi(y)x$. Hence

$$(\mu y) x = \mu (y \cdot x) \quad \forall \mu, y \in \Gamma, x \in X$$

In addition to the previous hypotheses on $f$ assume that $\Gamma$ acts on both $X$ and $y$

is equivariant, i.e.

$$(3) \ f(\gamma x, v) = y \cdot f(x, v) \quad \forall (x, v) \in U, y \in \Gamma$$

and $x_0$ is invariant, i.e.

$$y x_0 = x_0 \quad \forall y \in \Gamma$$

Then the Frechet derivative of (3) w.r.t. $x$ gives:

$$D_x f(yx, v) y = y D_x f(x, v) \quad \forall y \in \Gamma$$

where chain rule and $y \in B(X, X) \Rightarrow Dy = y$

have been used. So at $x_0 = yx_0$:

$$D_x f(x_0, v_0) y = y \cdot D_x f(x, v) \quad \forall y \in \Gamma$$
i.e. \( \ker \partial_x f(x_0, \lambda_0) \) commutes with the action.

Consequently:

\[ \gamma \cdot \ker \partial_x f(x_0, \lambda_0) \subseteq \ker \partial_x f(x_0, \lambda_0) \quad \forall \gamma \in \Gamma \]

and

\[ \gamma \cdot \text{range} \partial_x f(x_0, \lambda_0) \subseteq \text{range} \partial_x f(x_0, \lambda_0) \quad \forall \gamma \in \Gamma \]

Moreover \( x_2 \) and \( y_2 \) on page 1 can also be chosen invariant under the action of \( \Gamma \) which implies:

\[ P_0 y_2 = y_2 P_0 \quad \text{and} \quad Q_0 y_2 = y_2 Q_0 \quad \forall y_2 \in \Gamma \]

i.e. the projections are equivariant.

All these now imply the C map:

\[ x_2 : U_4 \rightarrow X_2 \quad U_4 \subset X_2 \times \Lambda \text{ open} \]

obtained on page 2, is equivariant:

\[ x_2 (\gamma x_1, \lambda) = \gamma x_2 (x_1, \lambda) \quad \forall \gamma \in \Gamma \]

and the reduced equation:

\[ c(x_1, \lambda) = (\text{Id} - Q)/(x_1 + x_2(x_1, \lambda), \lambda) = 0 \]

is equivariant, i.e.

\[ c(\gamma x_1, \lambda) = \gamma \cdot c(x_1, \lambda) \quad \forall \gamma \in \Gamma \]
Bifurcations in HLS

Recall our Schrödinger example:

\[ F : H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R} \to L^2(\mathbb{R}^n, \mathbb{C}) \]

\[ F(\Psi, E) = (-\Delta + V + E)\Psi - \gamma |\Psi|^2 \Psi \]

is \( C^1 \) if \( H^2, L^2 \) are viewed as real Banach spaces.

We want (3) \( F(\Psi, E) = 0 \) and we know

\[ F(0, E) = 0 \quad \forall E \in \mathbb{R}. \]

\[ D_\Psi(0, E) = -\Delta + V + E \]

**Lemma (Spectrum of \(-\Delta + V\))**

\[ (-\Delta + V + E) : H^2 \to L^2 \]

is an isomorphism for all \( E \in \mathbb{C} \setminus (-\infty, 0] \)

except maybe a countable set

\[ E_1 > E_2 > E_3 \ldots > 0 \]

on which

\[ 0 < \dim \ker (-\Delta + V + E) < \infty \]

Moreover, \( \exists ! \psi_0 \in H^2, \|\psi_0\|_2 = 1, \psi_0 > 0 \)

such that

\[ \ker (-\Delta + V + E_i) = \{ a\psi_0 \mid a \in \mathbb{C} \} \]

i.e., \( \dim \ker (-\Delta + V + E) = 1 \) over the complex field.
Lyapunov–Schmidt Reduction at \((0, E)\):

\[ H^2 = \text{Span}_{\mathbb{C}} \{ \psi_0, i \psi_0 \} \oplus (\psi_0^* \cap H^2) \]

(the orthogonal complement is with respect to the complex scalar product)

**Claim**: \(\text{Range } (-\mathbb{I} + V + E) = \psi_0^*\)

Hence on the co-domain we will use the decomposition

\[ L^2 = \psi_0^* \oplus \text{Span}_{\mathbb{C}} \{ \psi_0, i \psi_0 \} \]

Let \(P_1 : L^2 \to \psi_0^*\) be the orthogonal projection.

(3) is equivalent to:

\[
\begin{cases}
P_1 F(a, \psi_0 + i a_2 \psi_0 + h, E) = 0 \\
< \psi_0, F(a, \psi_0 + i a_2 \psi_0 + h, E) > = 0
\end{cases}
\]

**Proof of Claim**: By symmetry of \(-\mathbb{I} + V + E)\:

\[
< \psi_0, (-\mathbb{I} + V + E_1) \psi_0 > = < (E + V + E_1) \psi_0, \psi_0 >
\]

\[ = 0 \quad \forall \psi \in H^2 \]

So \(\text{Range } (-\mathbb{I} + V + E) \subseteq \psi_0^*\). To show equality we use the following general result:

**Remark (Spectral projections)**: If \(X\) is Bounded \(X(\mathbb{C}) \subseteq X\) is dense and
$A: (\mathcal{A}) \to X$

is a linear closed operator (i.e. the graph of $A$ is closed in $X \times X$) then $\sigma(A)$ is open in $\sigma(K)$ and

$$R: \sigma(A) \to B(X,X), \quad R(\lambda) = (A - \lambda I_x)^{-1}$$

is continuous and satisfies:

$$R(\lambda) - R(\mu) = (\lambda - \mu) R(\lambda) R(\mu)$$

hence $R$ is actually holomorphic.

Moreover if $\lambda \in \sigma(A)$ is isolated, i.e.

$$\exists \varepsilon > 0 :$$

$$\{ \mu \in C \mid |\mu - \lambda| < \varepsilon \} \cap \sigma(A) = \{ \lambda \}$$

then, $\forall 0 < \varepsilon < \varepsilon$ the integral

$$P_\lambda = \frac{1}{2\pi i} \oint_{|\mu - \lambda| = \varepsilon} (A - \mu I_x)^{-1} \, d\mu$$

is independent of $\varepsilon$ and defines $P_\lambda : X \to X$ with the properties:

1° $P_\lambda$ is linear, $P_\lambda^2 = P_\lambda$, i.e. $P_\lambda$ is a projection

2° $P_\lambda$ is continuous, range $P_\lambda$ and ker $P_\lambda$ are both closed

range $P_\lambda \oplus$ ker $P_\lambda = X$

range $P_\lambda \subseteq \sigma(A)$, $A(\text{range } P_\lambda) \subseteq \text{range } P_\lambda$

ker $P_\lambda$ is dense in ker $P_\lambda$; $A(\text{range } P_\lambda) \cap \text{ker } P_\lambda \subseteq \text{ker } P_\lambda$
\[ \mathcal{K}(A | \ker v_n) = \{ \lambda \gamma \} \]

\[ \lambda \not\in \mathcal{K}(A | \ker v_n) \]

Moreover, if \( v_n \) is finite dimensional then

\[ \exists \gamma \in (A - \lambda I)^{-1} \ker v_n = 0 \]

This general result applies to our case with

\[ V(A) = H^2, \quad X = L^2, \quad A = -\Delta + V, \quad \gamma = -E \]

and we have

\[ \ker v_n = v_0^{-1}, \quad \ker v_n = \{ a v_0 | a \in \mathbb{C} \} \]

because:

\[ V(A) \gamma v_0 = -\frac{1}{2\pi i} \int_{|\nu - \lambda| = \gamma} (-\Delta + \nu)^{-1} \gamma v_0 d\lambda \]

\[ = -\frac{1}{2\pi i} \int_{|\nu - \lambda| = \gamma} \frac{(-\Delta + \nu)^{-1} (-\Delta + \nu - \lambda) v_0}{\nu - \lambda} d\lambda \]

\[ = v_0 \]

and for any \( \psi \in L^2 \)

\[ \langle \psi_0, v_n \psi \rangle = -\frac{1}{2\pi i} \int_{|\nu - \lambda| = \gamma} \langle (-\Delta + \nu - \lambda) \psi_0, (-\Delta + \nu - \lambda)^{-1} \psi \rangle d\lambda \]

\[ = \langle \psi_0, \psi \rangle \left(-\frac{1}{2\pi i}\right) \int_{|\nu - \lambda| = \gamma} \frac{d\lambda}{\nu - \lambda} = \langle \psi_0, \psi \rangle \]

In particular
\[- E \otimes \mathcal{J}( - \gamma + V \mid \psi^+ \wedge H^2 ) \text{ i.e.} \]

\[- ( A + V + E ) : \psi^+ \wedge H^2 \rightarrow \psi^+ \text{ is an isomorphism.} \]

Now, by the standard duality - Schur's reduction, (3.1) gives \( A, F > 0 \) and the \( C^1 \) map

\[ h = h( a_1, a_2, E ) \text{ on } |B - E| < F, \quad |(a_1, a_2)| < \sigma. \]

For simplicity, we will identify:

\[ (a_1, a_2) = a_1 + ia_2 = a \in \mathbb{C}. \]

In order to solve (3.2) we need some properties of \( h \):

\[ h(0, E) = 0 \quad \text{(since } h \text{ is unique and } F(0, E) = 0 \text{).} \]

\[ h( e^{i\Theta} a, E ) = e^{i\Theta} h( a, E ), \]

\[ h( \bar{a}, E ) = \overline{h( a, E )} \]

hence \( h( a, E ) \) is real valued for real \( a \), \( |a| < \delta \).

This is because (3.1) inherits the (gauge) symmetries of (3.1):

\[ F( e^{i\Theta} \psi, E ) = e^{i\Theta} F( \psi, E ), \]

\[ F( \bar{\psi}, E ) = \overline{F( \psi, E )} \]

So, if \( (a, \psi + h( a, E ), E) \) is a sol of (3.1) then
\[ P_\perp P (e^{i \Theta} a \psi_0 + e^{i \Theta} w(a, E), E) = e^{i \Theta} P_\perp P (a \psi_0 + w(a, E), E) = 0 \]

i.e.

\[ (e^{i \Theta} a \psi_0 + e^{i \Theta} w(a, E), E) \text{ is a slu}. \]

By uniqueness \( u(e^{i \Theta} a, E) = e^{i \Theta} u(a, E) \).
A similar argument shows \( v(\overline{a}, E) = \overline{u}(a, E) \).

(3.1) can be written:

\[ (-a + V + E) u + y \mu \left[ a \psi_0 + w \right] (a \psi_0 + w) = 0 \]

or

\[ u = y (-a + V + E)^{-1} \mu \left[ a \psi_0 + w \right] (a \psi_0 + w) \]

Now, since \( |a| < \delta \):

\[ u(0, E) = 0 \]

and

\[ \frac{\partial u}{\partial a} (0, E) = y (-a + V + E)^{-1} \mu \left[ a \psi_0 + w \right] (a \psi_0 + w) \bigg|_{a=0} = 0 \]

\[ = 0 \]

so \( u(a, E) = O(|a|) \) and plugged in above we get:

\[ u(a, \lambda) = O(|a| \lambda^3). \]

Then for \( a \in \mathbb{R}; |a| < \delta \):

\[ \frac{\partial u}{\partial E} = \left[ -a + V + E + y (3a^2 \psi_0^4 + 6a \psi_0 w + 3w^2) \right]^{-1} \]

\[ \frac{\partial u}{\partial E} = O(|a| \lambda^3). \]
We are now ready to analyze (3.2):

\[(E - E_1)a + \langle \psi_0, \gamma | a\psi_0 + \alpha | \psi_0 + \alpha \rangle \rangle = 0\]

where \(\alpha = \alpha(a, E)\), \(a \in \mathbb{C}\), \(|a| < \frac{1}{\sqrt{2}} |E - E_1| \sqrt{\varepsilon}\)

\[a = 0\] is a sol for all \(E\) since \(\alpha(0, E) = 0\)

We already knew about the solution since \(\alpha(0, E) = 0\) for all \(E\).

We now look for sols \(a \neq 0\). Divide by \(a\)

\[(E - E_1) + \gamma \langle \psi_0, (a\psi_0 + \alpha | \psi_0 + \alpha \rangle \rangle = 0\]

Consider \(a = \alpha e^{i\theta}\) and use

\[\alpha(\alpha e^{i\theta}, E) = \alpha(\alpha e^{i\theta}, E) e^{i\theta} \alpha(\alpha, E)\]

to get the equivalent equation:

\[(x) \quad 0 = E - E_1 + \gamma \langle \psi_0, (\alpha \psi_0 + \alpha | \psi_0 + \alpha \rangle \rangle \cdot \left(\frac{\psi_0 + \alpha}{\alpha}\right)\]

Let

\[G(a, E) = \begin{cases} E - E_1 + \gamma \langle \psi_0, \left(\alpha \psi_0 + \alpha | \psi_0 + \alpha \rangle \right) \cdot \left(\frac{\psi_0 + \alpha}{\alpha}\right) & \text{for } a \neq 0 \\ 0 & \text{for } a = 0 \end{cases}\]

\((x)\) is the restriction to \(a > 0\) of

\[(m) \quad G(a, E) = 0\]

Now, we can check that \(G : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \rightarrow \mathbb{R}\)

is \(C^1\) and

\[G(0, E_1) = 0\]

\[\frac{\partial G}{\partial E}(0, E_0) = 1 + \gamma \langle \psi_0, \frac{3(\alpha \psi_0 + \alpha | \psi_0 + \alpha \rangle^2}{\alpha} \rangle \right|_{\alpha = 0} = 1\]
By the implicit function theorem there is a unique curve of solutions for \( \gamma \):
\[
E = E(\alpha) \quad \alpha \in \mathbb{R}, \quad 1\alpha < \delta < \delta
\]

Translated back to the complex equation \( \gamma \), we have a unique solution:
\[
E = E(1\alpha) \quad \alpha \in \mathbb{C}, \quad 1\alpha < \delta
\]
and
\[
\gamma = \alpha y_0 + u(\alpha, E(1\alpha)) \quad \alpha \in \mathbb{C}, \quad 1\alpha < \delta
\]
the latter being the second branch of the \( \gamma \) of
\[
F(\gamma, E) = 0 \quad \text{besides} \quad \gamma = 0
\]

Conclusion: Eq. (1) has a unique branch of nontrivial bounded states bifurcating from a simple negative eigenvalue of \(-\lambda + \nu\).

Remarks: The type of bifurcation in (5) means in the reduced eq. (3.2) is called a pitchfork bifurcation. We can draw the following bifurcation diagram:

![Bifurcation Diagram](image)

References:

Secondary bifurcations along the non-birefringent ground state branch.

How much can we extend the branch (two-dimensional manifold) of non-birefringent ground states given by the above bifurcation from \( 0, E_1 \)?

The answer depends on the sign of the nonlinearity.

If \( \gamma > 0 \) (repelling nonlinearity), the branch can be extended for all \( 0 < E < E_1 \), but what exactly happens when \( E = 0 \) is still an open problem.

If \( \gamma < 0 \) (attracting nonlinearity), the answer seems to depend on the number of critical points of the potential \( V \). For example, in 1 space dimension, \( n = 1 \), if \( V(x) \) is even and strictly monotonous for \( x > 0 \), hence it has a unique critical point, the global minima at \( x = 0 \) then the ground state branch can be extended for all \( E > E_1 \), see

However, if $V(x)$ is even but has two distinct local minima (and a local maximum at $x=0$) then a bifurcation must occur on the non-trivial branches of ground states, see


To understand why let's look at:

$$D_{\psi} \left[ \psi_E, E \right] \sum_1^{\psi_1 + i\psi_2} = (-\Delta + V + E) \left[ \psi_1 + i\psi_2 \right]$$

$$+ 2\gamma \left| \psi_E \right|^2 \sum_1^{\psi_1 + i\psi_2} \psi_1 + i\psi_2 \left( -\Delta + V + E \right) \left[ \psi_1 - i\psi_2 \right] =$$

$$\begin{bmatrix}
-\Delta + V + E + 2\gamma \left| \psi_E \right|^2 + \gamma \Re \psi_2^2 , -\gamma \Im \psi_1^2 \\
\gamma \Im \psi_1 \psi_E^2 , -\Delta + V + E + 2\gamma \left| \psi_E \right|^2 - \gamma \Re \psi_2 \psi_2
\end{bmatrix} \begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix}$$
provided

Your $\Psi$ on the ground state $|E_1\rangle$ can be chosen real valued, hence

$$\Phi (\Psi, E) \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} L_+ \left( \Psi_2 \right) & 0 \\ 0 & L_- \left( \Psi_1 \right) \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$

with

$$L_+ = -\Delta + V + E + 3\gamma \Psi_2^2$$

$$L_- = -\Delta + V + E + \gamma \Psi_1^2$$

One can show, for $E$ close to $E_1$:

$$\ker L_- = \text{span} \{ \Psi_1 \}$$

$$\ker L_+ = \{ 0 \}$$

But $i \Psi_2 \in \ker \Phi (\Psi, E)$ because it is the tangent vector in the direction of rotations which generate the complex valued $e^{it\Psi}$ ground state.
from the real valued $t$ "shooting out" notations we can uniquely extend the branch via implicit function theorem as long as:

$$\ker L^+ = \{0\}$$

Using regular perturbation theory and comparison principle we have for $E$ close to $E_1$

$$n = E - E_1$$

To show that the second $c$-value of $L^+$ crosses zero one can use two arguments:

1° in the limit when the distance $L$ between the wells of the potential (see figure above) goes to $\infty$, it is known that

$$\lambda_1(0, L) \to 0; \lambda_1(L) > 0$$

and one can calculate

$$\frac{\partial \lambda_1(0, L)}{\partial L} \to -1/2 < 0$$

hence, for $L$ sufficiently large, $\lambda_1(E - E_1, L)$ must cross zero at $E = E_1$ close to $E_1$, see
or (better) Corollary 2 in


2) If the distance between the walls is not sufficiently large, one assumes that $\lambda_+(E)$ does not cross zero. Then proves that thebranche can be extended for all $E > E_1$, i.e. it does not blow up at finite $E$, and shows that as $E \to \infty$, $L_+(E)$ must have at least two negative $\omega$-values and obtain a contradiction. For details, see: