3. Calculus in Banach spaces, the Implicit Function Theorem and applications to Bifurcation Theory

3.1. Calculus in Banach spaces

Let \( X, Y \) be two Banach spaces over \( \mathbb{R} \) or complex numbers and \( B(X,Y) \) the space of linear, bounded operators from \( X \) into \( Y \).

Let \( F : U \rightarrow Y \), \( U \subseteq X \) open be a map.

**Definition:** \( F \) has Gâteaux derivative at \( u \in U \) in the direction \( x \in X \) if and only if:

\[
\lim_{\varepsilon \to 0} \frac{F(u+\varepsilon x) - F(u)}{\varepsilon}
\]

exists and is finite.

Notation: \( dF(u)[x] = \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon x) - F(u)}{\varepsilon} \)

**Definition:** \( F \) is Gâteaux differentiable at \( u \in U \) if and only if it has Gâteaux derivative in any direction.

**Remark:** If \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) then

\[
dF(x_1, x_2)[(1,0)] = \frac{\partial F}{\partial x_1}(x_1, x_2)
\]
Def. F is Fréchet differentiable at \( u \in U \) if there exists \( A \in B(X, Y) \) such that:

\[
\lim_{x \to 0} \frac{\|F(u + x) - F(u) - Ax\|}{\|x\|} = 0
\]

**Variation:** \( A = DF(u) \) is called the Fréchet derivative.

**Remark 2**: Fréchet differentiability at a point is less restrictive than Gâteaux differentiability, and for functions \( F: \mathbb{R}^m \to \mathbb{R}^n \) it coincides with differentiability at a point.

**Theorem**: If \( F \) is Fréchet differentiable at \( u \in U \) (with Fréchet derivative \( A \)) then \( F \) is Gâteaux differentiable at \( u \) and

\[
A x = DF(u)[x] \quad \forall x \in X
\]

**Proof**: We have:

\[
\lim_{E \to 0} \frac{F(u + E x) - F(u) - A x}{E} = 0
\]

Passing to the limit where \( E \to 0 \) we get

\[
\lim_{E \to 0} \frac{E(F(u + E x) - F(u))}{E} = A x \quad \text{QED}
\]

**Remark 3**: Gâteaux differentiable \( \Rightarrow \) Fréchet differentiable. Example:

\( f: \mathbb{R}^2 \to \mathbb{R} \)

\[
f(x, y) = \begin{cases} 
x^2 y^2 \quad &\text{if } (x, y) \neq (0, 0) \\
0 \quad &\text{if } (x, y) = (0, 0)
\end{cases}
\]

\[
Df(0, 0)[(a, b)] = \lim_{E \to 0} \frac{1}{E} \left( \frac{E a^2}{2} \right) = \begin{cases} 
a^2 \quad &E \neq 0 \\
0 \quad &E = 0
\end{cases}
\]

\( 0, b = 0 \)
\[ f(0,0) : \mathbb{R}^2 \to \mathbb{R} \text{ is not linear so } f \text{ cannot be Fréchet differentiable.} \]

**Theorem:** If \( F \) is Fréchet differentiable at \( u \in U \) then \( F \) is continuous at \( u \).

**Proof:** Let \( \{x_n\} \subseteq U \), \( \lim_{n \to \infty} x_n = u \). Then

\[
F(x_n) - F(u) = F(u + x_n - u) - F(u) = \\
= F(u + x_n - u) - F(u) - A(x_n - u) + A(x_n - u) \\
= \|x_n - u\| \cdot \frac{F(u + x_n - u) - F(u) - A(x_n - u)}{\|x_n - u\|} \\
+ A(x_n - u)
\]

Since \( A \) is bounded, \( A(x_n - u) \to 0 \) as \( n \to \infty \) hence

\[
\lim_{n \to \infty} (F(x_n) - F(u)) = 0.
\]

Since \( x_n \to u \) we see

arbitrarily \( F \) is continuous at \( u \in U \). QED

**Remark 4:** For \( f : \mathbb{R}^2 \to \mathbb{R} \)

\[
f(x, y) = \begin{cases} \frac{x^3 y}{x^6 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
\]

We have \( df(0,0) [x, y] = 0 \) \( \forall (x, y) \in \mathbb{R}^2 \)

hence the Gâteaux derivative is linear and bounded but \( f \) is still not Fréchet differentiable at \( (0,0) \) because it is not continuous; \( \lim_{n \to \infty} f(\frac{1}{n}, \frac{1}{n^2}) = \frac{1}{2} \neq f(0,0) \)

**Theorem:** Let \( U \subseteq \mathbb{R}^n \) with \( U \) open.

Assume \( \frac{dF}{dx}(u) \in B(x,y) \) \( \forall u \in V \) and \( dF(\cdot) : V \to B(x,y) \) is continuous at \( U \). Then
F is Fréchet differentiable at v.

Proof: We first need an extension of the mean value theorem:

Lemma: If \( u, v \in X \) and \( F \) has Fréchet derivative in direction \( v - u \) for all the points in the segment:

\[
[v, u] = \{ x \in X \mid x = (1-t)u + tv \quad \text{for some} \quad 0 \leq t \leq 1 \}
\]

then \( \exists x \in [v, u] \setminus \{ v, u \} \) such that

\[
\| F(u) - F(v) \| \leq \| dF(x) [v - u] \|.
\]

Proof of Lemma: Let \( y^* \in Y^* \) such that

\[
\| y^* \| = 1 \quad \text{and} \quad y^* (F(u) - F(v)) = \| F(u) - F(v) \|. \]

(such a functional exists via Hahn-Banach Theorem)

Let \( f : [0,1] \to \mathbb{R} (or C) \) \( f(t) = y^* F((1-t)u + tv) \).

Now, via direct calculation, \( f \) is differentiable on \([0,1] \):

\[
f'(t) = y^* dF((1-t)u + tv) [v - u].
\]

Hence, via mean value theorem (for \( f, f' \) on \([0,1] \):

\[
f(1) - f(0) = f'(t) \quad (\text{in } f \text{ real valued})
\]

\[
|f(1) - f(0)| \leq |f'(t)| \quad (\text{for } f \text{ complex valued})
\]

You would now use now

\[
|y^* F(u) - y^* F(v)| \leq |y^* dF((1-t)u + tv) [v - u]|.
\]

which implies
\[
\| F(v) - F(u) \| \leq \| y \| \| dF((1-t)v + tv) [v-u] \| \\
\leq \| dF ((1-t)u + tv) [v-u] \|
\]

The lemma is completely proven.

Back to the theorem: \( \exists \delta > 0 \) such that:
\[
B(u, \delta) = \{ x \in X \mid \| x - u \| < \delta, y \in V \}
\]

Consider
\[
G : B(u, \delta) \rightarrow V \\
G(v) = F(v) - F(u) - dF(u) [v-u].
\]

A direct calculation shows that \( G \) is \( \mathcal{C}^1 \) differentiable on \( B(u, \delta) \) with:
\[
\frac{dG(v)}{dx} = dF(v)(x) - dF(u)(x)
\]

Here we used the linearity of \( dF(u) \).

Apply previous Lemma on Segments \( [u,v] \subset B(u,\delta) \)

\( \Rightarrow \forall \delta \leq B(u, \delta) \exists t_v \in (0,1) \) such that:
\[
\| G(v) - G(u) \| \leq \| dG((1-t_v)u + t_v v) [v-u] \|
\]

\( \Rightarrow \| F(v) - F(u) - dF(u) [v-u] \| \leq \| (dF((1-t_v)u + t_v v) - dF(u)) [v-u] \| \\
\leq \| dF((1-t_v)u + t_v v) - dF(u) \| \| v-u \|
\]

Hence for \( v \rightarrow u \):
\[
\| F(v) - F(u) - dF(u) [v-u] \| \leq \| dF((1-t_v)u + t_v v) - dF(u) \| \| v-u \|
\]

Now \( \lim_{v \rightarrow u} (1-t_v)u + t_v v = u \) and by
continuity of $dF: \mathcal{B}(U,\varepsilon) \to \mathcal{B}(X,Y)$ at $u$ we deduce

\[
\lim_{v \to u} \frac{\|F(v) - F(u) - dF(u)(v-u)\|}{\|v-u\|} = 0
\]

i.e. $F$ is Fréchet differentiable at $u$. \qed

Remark 5. Both Gâteaux and Fréchet differentiable functions verify:

(i) \[ d(F + G)(u) = dF(u) + dG(u) \]
\[ D(F + G)(u) = DF(u) + DG(u) \]

(ii) \[ d(\lambda F)(u) = \lambda dF(u) \quad \forall \lambda \in \mathbb{R}(or \mathbb{C}) \]
\[ D(\lambda F)(u) = \lambda DF(u) \quad \forall \lambda \in \mathbb{R}(or \mathbb{C}) \]

In addition the Fréchet differential satisfies

(iii) \[ D(F \circ G)(u) = DF(G(u)) \circ DG(u) \]

but this property may fail for Gâteaux differentials:

\[ d(F \circ G)(u) \text{ might not exist} \]

even where \[ dF(G(u)) \text{ and } dG(u) \text{ exist!} \]

Remark 6. Higher order derivatives can be defined. For example, if

\[ dF(u)[v] \text{ exists for } u \in (U_0 - \varepsilon_0 W, U_0 + \varepsilon_0 W), \varepsilon_0 > 0 \]

then:
\[
\frac{d^2 F(u_0)[\omega][\nu]}{\nu} = \lim_{\epsilon \to 0} \frac{dF(u_0+\epsilon \omega)[\nu]-dF(u_0)[\nu]}{\epsilon}
\]

Also, if \( D^2 F(u) \in \mathcal{B}(X, Y) \) exists in open set \( U \) then for \( u_0 \in U \):

\[
D^2 F(u_0) = D(D^{-1}F(\cdot))(u_0)
\]

provided the latter exists, in other words

\[
\exists \ D^2 F(u_0) \in \mathcal{B}(X, \mathcal{B}(X, Y)) \text{ such that:}
\]

\[
\lim_{w \to 0} \frac{||DF(u_0+xw)-DF(u_0)-D^2 F(u_0)[\omega][\nu]||}{||w||} = 0
\]

Write that if \( D^2 F(u_0) \) exists then

\[
D^2 F(u_0)[\omega][\nu][\nu] = D^2 F(u_0)[\omega'][\nu'][\nu']
\]

Inductively,

\[
D^{2n} F(u_0) = D(D^{2(n-1)}F(\cdot))(u_0) \in \mathcal{B}(X, \mathcal{B}(X, \ldots, \mathcal{B}(X, Y)))
\]

and we say that \( F \in C^0(U) \) if

\[
u \mapsto D^{2n} F(u) \text{ is continuous.}
\]

Remark: Integral calculus can also be recovered.

For example if \( D(F(\cdot))[\omega][\nu] \) is continuous on the segment \([u, \nu]\) then

\[
F(\nu)-F(u) = \int_0^1 dF((1-t)u + tv)[\nu-\nu] \; dt
\]
Example 1 \( F: H^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C}) \)

\[ F(\phi) = (-\Delta + V)\phi + \gamma |\phi|^2 \phi \]

where \( \gamma \in \mathbb{R} \) and \( V \in L^\infty(\mathbb{R}^n, \mathbb{R}) \).

- \( F \) is well defined:

\[ \forall \phi \in L^2 \text{ because } V \in L^\infty \text{ and } \phi \in L^2 \]

\[ |\phi|^2 \phi \in L^2 = \phi \in L^6 \text{ true because, for } \]

\[ n = 1, 2, \ldots, 6 \text{, we have, by Sobolev Embedding } \]

\[ H^2(\mathbb{R}^n) \hookrightarrow L^6(\mathbb{R}^n) \]

- Gateaux derivative

\[ dF(\phi_0)[\psi] = \lim_{\varepsilon \to 0} \frac{F(\phi_0 + \varepsilon \psi) - F(\phi_0)}{\varepsilon} \]

\[ = (-\Delta + V)\psi + 2\gamma |\phi_0|^2 \psi + \lim_{\varepsilon \to 0} \frac{\gamma |\phi_0|^2 \psi}{\varepsilon} \]

where \( \overline{\psi} \) = complex conjugate of \( \psi \in \mathbb{C} \)

The limit does not exist for \( \varepsilon \in \mathbb{C} \) but only for \( \varepsilon \in \mathbb{R} \). We need to consider the Hilbert spaces

\[ H^2(\mathbb{R}^n, \mathbb{C}) \text{ and } L^2(\mathbb{R}^n, \mathbb{C}) \]

over the field \( \mathbb{R} \). In this case:

\[ \langle f, g \rangle = \text{Re} \int_{\mathbb{R}^n} \overline{f(x)}g(x) \, dx \]
and no change in the definition of the norm is needed. Now:

$$\mathcal{L}F(\psi_0)[\psi] = (-N^2 + N^2)\psi + 2N^2|\psi_0|^2\psi + N^2\psi$$

and $\mathcal{L}F(\psi_0): H^2 \to L^2$ is linear, bounded and depends continuously on $\psi_0$

$\Rightarrow$ Fréchet derivative exists:

$$DF(\psi_0)[\psi] = (-N^2 + N^2 + 2N^2|\psi_0|^2 + N^2\psi_0)$$

and depends continuously on $\psi_0$, i.e. $F$ is $C^1$. 