Lecture 4

2. Banach and Hilbert Spaces

2.1 General Properties of Banach spaces and examples

2.2 General Properties of Hilbert spaces and examples (see Lecture 5)

2.3 Banach spaces and examples

Def (Normed spaces) A vectorial space $V$ together with a function $|| \cdot || : V \rightarrow [0, \infty)$ such that:

(i) $|| v || = 0$ if and only if $v = 0$
(ii) $|| av || = |a| || v ||$ for all $a \in \mathbb{R}$
(iii) $|| v + w || \leq || v || + || w ||$ for all $v, w \in V$

is called a normed space

Example 1. For $1 \leq p < \infty$ and $x \in \mathbb{R}^n$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$|| x ||_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

defines a norm on $\mathbb{R}^n$, so $(\mathbb{R}^n, || \cdot ||_p)$ are normed spaces. Checking (i) and (ii) is trivial, but for (iii) one needs Minkowski's inequality (see Handout).

2. For $x \in \mathbb{R}^n, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$|| x ||_\infty = \max \{ |x_1|, |x_2|, \ldots, |x_n| \}$$

defines a norm on $\mathbb{R}^n$, so $(\mathbb{R}^n, || \cdot ||_\infty)$ is a normed space.
**Example 1** Any normed space is a metric space with canonical metric

\[ d(u, v) = \| u - v \| \quad \forall (u, v) \in V \times V \]

which is invariant under translations:

\[ d(u + w, v + w) = d(u, v) \quad \forall u, v, w \in V \]

and homogeneous:

\[ d(\lambda u, \lambda v) = |\lambda| d(u, v) \quad \forall u, v \in V, \lambda \in \mathbb{R} (\neq 0) \]

Topological notions (open, closed, compact subsets, convergence, continuity) follow from this canonical metric. In particular:

**Def (Bounded sets)** A subset \( U \subseteq V \) is bounded if

\[ \exists M > 0 \text{ such that } \| u \| \leq M \quad \forall u \in U, \]

and:

**Def (Banach spaces)** A Banach space is a normed space which is complete with respect to the canonical metric, i.e. any (Cauchy) sequence \( \{ u_n \} \subseteq V \) with the property:

\[ \forall \varepsilon > 0 \exists N \in \mathbb{N} : \| u_n - u_m \| < \varepsilon \quad \forall n, m \geq N \]

is convergent, i.e. \( \exists v \in V \) such that:

\[ \forall \varepsilon > 0 \exists N \in \mathbb{N} : \| u_n - v \| < \varepsilon \quad \forall n \geq N. \]
Remark 3. If \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent norms, then their associated metrics are equivalent, see Lecture 2. In particular, open (closed, compact, bounded) sets in one norm are open (closed, compact, bounded) in the other norm, and convex, respectively, convergent sequences in one norm are convex, respectively, convergent in the other, etc.

Remark 4. If \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) are normed spaces then
\[
F : U \to Y, \quad U \subseteq X
\]
is continuous at \( x_0 \in U \) iff
\[
\exists \delta > 0 \quad \forall x \in U : \| x - x_0 \|_X < \delta \implies \| F(x) - F(x_0) \|_Y < \varepsilon.
\]
or, equivalently, if \( F \) transforms any sequence in \( X \) convergent to \( x_0 \) into a sequence convergent to \( F(x_0) \). Mappings that are continuous at any point in the domain are called continuous.

Note that a map is continuous if and only if preimages of open (closed) sets are open (closed). Also note that a continuous map transforms compact sets into compact sets.

**Linear, Continuous Mappings.**

**Def.** If \( X, Y \) are vector spaces over the same field \( K \) (\( \mathbb{R} \) or \( \mathbb{C} \)) then the map \( F: X \to Y \) is called linear if:

\[
F(x_1 + x_2) = F(x_1) + F(x_2) \quad \forall x_1, x_2 \in X
\]
\[
F(cx) = cF(x) \quad \forall x \in X, c \in K
\]

**Theorem.** If \( (X, \| \cdot \|_X), (Y, \| \cdot \|_Y) \) are normed spaces over the same \( K \) (\( K = \mathbb{R} \) or \( K = \mathbb{C} \)), and \( F: X \to Y \) is linear then the following are equivalent:

\( i \)  \( F \) is continuous

\( ii \)  \( F \) is continuous at \( 0 \)

\( iii \)  \( \exists L > 0 \) such that

\[
\|F(x)\|_Y \leq L \|x\|_X \quad \forall x \in X.
\]

**Remark.** A function satisfying \( iii \) is sometimes called bounded. However, \( ii \) really means Lipschitz because, for all \( x, y \in X \):
\[ L_y (F(x), F(y)) = \| F(x) - F(y) \|_Y \leq L\| x-y \|_X = L \| x, y \| \\
\]

**Proof of Theorem**

(i) \( \Rightarrow \) (ii) follows.

(ii) \( \Rightarrow \) (iii). By definition, for \( \varepsilon = 1 \), \( \exists \delta > 0 \):
\[ \| F(x) - F(0) \|_Y < \varepsilon \quad \forall x : \| x - 0 \|_X < \delta \]

By linearity, \( F(0) = 0 \). Now for any \( x \in X \), consider
\[ \tilde{x} = \frac{x}{\| x \|_X} \quad \delta = \frac{\delta}{\| x \|_X} \]  
Then
\[ \| \tilde{x} \|_X = \frac{\delta}{\| x \|_X} \quad \frac{\| \tilde{x} \|_X}{\| x \|_X} = \frac{\delta}{\| x \|_X} < \delta \]

\[ \Rightarrow \quad \| F(\tilde{x}) \|_Y = \| F \left( \frac{\delta}{\| x \|_X} x \right) \|_Y \]

by linearity
\[ \frac{\| F(\tilde{x}) \|_Y}{\| \tilde{x} \|_X} = \frac{\| F(\tilde{x}) \|_Y}{\| \tilde{x} \|_X} = \frac{\| F(\tilde{x}) \|_Y}{\| \tilde{x} \|_X} \]

\[ \Rightarrow \quad \| F(x) \|_Y \leq \frac{\varepsilon}{\delta} \| x \|_X \quad \forall x \in X \setminus \{0\} \]

The inequality also holds for \( x = 0 \) since \( F(0) = 0 \).

(know \( F(0) = F(0, 0_x) = 0 \)).

So we can choose \( L = \frac{\varepsilon}{\delta} \) and (iii) is proven.

(iii) \( \Rightarrow \) (i) Remove \( \delta \) spheres (iii) \( \Rightarrow \) \( F \) is Lipschitz.

\[ \Rightarrow \] \( F \) is continuous (this is a general result in metric spaces, see Lecture 2, but basically, for any \( \varepsilon > 0 \) one chooses \( \delta_x = \frac{\varepsilon}{L} \). QED
Remark 6. If \((X, \| \cdot \|)\) is a normed space over \(K = \mathbb{R}\) or \(\mathbb{C}\), then for each \(a \in K\):
\[
+ : X \times X \to X \quad + (u, v) = u + v
\]
\[
\cdot : X \to X \quad \cdot (u) = au
\]
are linear continuous functions. Here \(X \times X\) is a normed space with \(\| (u, v) \| = \| u \| + \| v \|\).
With these definitions, the projections are continuous, in particular \(+\) is continuous in each variable. Note that \(\cdot : (x, x) \to x \cdot (x, x) = x^2\) is not linear, but it is continuous when \(K = \mathbb{R}\) for the norm \(\| x \| = |x| + |x|\), and, consequently, it is also continuous in \(\mathbb{R}\).

Theorem (Banach space of linear continuous maps) \(\mathbb{F}_{xy}\). If \(X, Y\) are normed spaces over the same \(K = \mathbb{R}\) or \(\mathbb{C}\), consider the vector space:
\[
B(X, Y) = \{ F : X \to Y \mid F\text{ linear and continuous} \}
\]
and \(\| F \| = \sup \{ \| F(x) \|_y \mid x \in X, \| x \|_x \leq 1 \}\).
Then \(\| \cdot \|\) is a norm on \(B(X, Y)\). Moreover, if \((Y, \| \cdot \|_y)\) is Banach then \((B(X, Y), \| \cdot \|)\) is Banach.

Proof. (HINTS: use (iii)) In the presence of \(K\) to show that \(\| \cdot \| : B(X, Y) \to \mathbb{R}^+\) is well-defined (i.e., finite at all \(F \in B(X, Y)\)). Its norm properties follow from the corresponding properties of \(\| \cdot \|_y\).
For the Banach property use a similar argument to \(\ell^p, (\ell^2, \ell^1, \ell^\infty)\) complete, see Lecture 3.

Theorem (Closed graph) If \(X, Y\) are Banach and \(F : X \to Y\) is linear then:
\(F\) is continuous \(\iff\) \(\{ (x, F(x)) \mid x \in X, y \in Y \} \subset X \times Y\) is closed.
Duality in normed spaces

Let $X$ be a normed space (over $K = \mathbb{R}$ or $\mathbb{C}$). Then $B(X, K) = X^*$ is called the dual of $X$.

$$X^* = \{ x^* : X \to K \mid x^* \text{ linear and continuous} \}$$

and is a Banach spaces w.r.t. the norm:

$$\| x^* \| = \sup \{ | x^*(x) | \mid \| x \| \leq 1 \}$$

Theorem (Hahn-Banach) If $Y \subseteq X$ is a linear subspace of $X$ and $y^* : Y \to K$ is linear such that:

$$| y^*(y) | \leq p(y) \quad \forall y \in Y$$

for a given seminorm $p : X \to \mathbb{R}_+$, then

$$\exists x^* : X \to K \text{ linear, } x^*(y) = y^*(y) \quad \forall y \in Y \quad \text{ and}$$

$$\| x^* \|_Y = p(x) \quad \forall x \in X.$$  

Corollary If $X$ is normal, $Y \subseteq X$ and $y^* \in Y^*$ then

$$\exists x^* \in X^* : x^*_Y = y^* \quad \text{and} \quad \| x^* \|_Y = \| y^* \|_Y.$$  

If $X$ is normal and $x \in X$ then

$$\exists x^* \in X^* \setminus \{ 0 \} \text{ with } x^*(x) = \| x^* \|_Y \| x \|.$$  

If $X$ is normal, $Y \subseteq X$, $x \in X \setminus Y$ and

$$d = \text{dist} (x, Y) = \inf \{ \| x - y \|_Y \mid y \in Y \}$$

then

$$\exists x^* \in X^* \text{ such that } \| x^* \|_Y \leq 1, \quad x^*_Y(x) = d, \quad x^*_Y = 0 \quad \forall y \notin Y.$$
**Theorem (The double dual).** Since \((X^*, || \cdot ||_*)\) is a normed space, we can define its dual \((X^*)^* = X^{**}\).

**Remark 7.** There exists a canonical embedding:

\[ J : X \rightarrow X^{**} \]

\[ J(x) \text{ evaluated at } x^* \in X^* \text{ is } x^*(x), \text{i.e.} \]

\[ J(x)(x^*) = x^*(x), \forall x^* \in X^*. \]

\(J\) is linear, continuous and one-to-one (hence an embedding). Moreover:

\[ \| J(x) \|_{X^{**}} = \sup_{x^* \in X^*} \| J(x)(x^*) \| = \| x \| \]

\[ \| x \|_{X^*} \leq 1 \]

Since by second countability of Hahn–Banach Th., for a given \(x \in X\) one can construct \(x^* \in X^*\) :

\[ \| x^* \| = 1 \text{ and } x^*(x) = \| x \|. \]

**Remark 8.** Any normed space can be completed.

Indeed, take

\[ J(X) \subseteq X^{**} \text{ (w.r.t. } \| \cdot \|_{X^{**}} \text{ norm)} \]

Since \(X^{**}\) is Banach, \(J(X)\) will also be Banach.

**Definition (Reflexive Banach spaces).** \(X\) is called reflexive if \(J : X \rightarrow X^{**}\) is onto.

In this case \(J\) is an isomorphism and isometry between \(X\) and \(X^{**}\). Hence \(X\) is also Banach.
Theorem (embedding of dual spaces) Let $X, Y$ be normed spaces such that

$$X \xrightarrow{\text{dense}} Y$$

then

$$Y^* \xrightarrow{\text{dense}} X^*$$

If, in addition, $X$ is reflexive, then $Y^*$ is dense in $X^*$.

Proof. $X \xrightarrow{\text{dense}} Y$ means $\exists i : X \to Y$

linear continuous and injective. Dense embedding means $i(X)$ is dense in $Y$, i.e.

$$\overline{i(X)} = Y.$$

Consider $i^*(y^*) = y^o i X \xrightarrow{\text{i}} Y \xrightarrow{y^*} \mathbb{K}$

Then

$$i^* : Y^* \xrightarrow{\text{dense}} X^*$$

is well defined, i.e. $y^* i \in X^*$ which follows from $i$ and $y^*$ linear and continuous.

Moreover $i^*$ is linear: $\forall y_1^*, y_2^* \in Y^* \text{ and } \lambda_1, \lambda_2 \in \mathbb{K}$

$$i^*(\lambda_1 y_1^* + \lambda_2 y_2^*) = \lambda_1 i^*(y_1^*) + \lambda_2 i^*(y_2^*)$$

and continuous since for all $y^* \in Y^*$ we have:

$$\|i^*(y^*)\|_{\mathbb{K}} = \|y^* o i\|_X = \sup_{\|x\|_X \leq 1} |y^*(i(x))|

and for all $x \in X$:

$$\|i^*(y^*)\|_X \leq \|y^*\|_{\mathbb{K}} \|i(x)\|_X \leq \|y^*\|_{\mathbb{K}} - \|x\|_X$$
where $0 < L < \infty$ satisfying $\| i^*(y^*) \|_{\psi} \leq L \| y^* \|_{\psi}$ should exist because $i: X \to Y^*$ is linear and continuous. All in all we get:

$$\| i^*(y^*) \|_{\psi} \leq L \| y^* \|_{\psi} \quad \forall y^* \in Y^*$$

So $i^*: Y^* \to X^*$ is linear and continuous.

It is also injective (one-to-one) because if

$$i^*(y^*) = 0$$

then $y^*(i(x)) = 0 \quad \forall x \in X$. Let $Y_i = i(X)$

We have:

$$\forall y^* \in Y_i, \quad y^* = 0 \quad \Rightarrow \quad i^*(y^*) = 0$$

$$\forall y^* \in Y_i, \quad \overline{y}^* = y^*$$

$$\Rightarrow \quad Y_i = 0 \quad \Rightarrow \quad i^* \text{ is injective.}$$

We construct a linear, continuous, injective map from $Y^*$ to $X^*$ as follows:

$$Y^* \to X^* = i^*$$

For $i^*(Y^*) = X^*$ we argue by contradiction. Assume $\exists x_0^* \in X^* \setminus i^*(Y^*)$.

By the third corollary of Hahn-Banach Theorem, there exists $x_0^{**} \in X^{**}$ such that

$$\| x_0^{**} \|_{X^{**}} \leq 1$$

and

$$x_0^{**}(x^*) = d = \text{dist}(x_0^*, i^*(y^*)) > 0, \quad \frac{x_0^{**}}{i^*(y^*)}$$

By Hahn-Banach Theorem, there exists $x_0 \in X$ such that $\langle x^*, x_0 \rangle = x_0^{**}$, i.e.

$$x_0^{**}(x^*) = i^*(x_0^*) \quad \forall x^* \in X^*$$
In particular, \( x^*(x_0) = d > 0 \) and

\[ x^*(x_0) = 0 \quad \forall x^* \in \Gamma(y^*) = y^*oI, \quad y^* \in Y^* \]

i.e. \( y^*(i(x_0)) = 0 \quad \forall y^* \in Y^* \) which implies \( i(x_0) = 0 \) otherwise by the second corollary of Hahn-Banach \( \exists y^* \in Y^* \setminus \{0\} : \)

\[ y^*(i(x_0)) = \|y^*\| \|i(x_0)\| > 0. \]

Now \( i(x_0) = 0 \) implies \( x_0 = 0 \) since \( i \) is injective. \( x_0 = 0 \) implies \( x_0^* (x_0) = 0 \) which contradicts \( x_0^*(x_0) = d > 0 \).

The theorem is now completely proven.

**Hooch Topologies in Banach spaces**

Let \( X \) be a normed space over \( \mathbb{K} (\mathbb{R} \text{ or } \mathbb{C}) \)

\[ X^+ = \{ x^* : X \to \mathbb{K} \mid x^* \text{ linear} \} \]

and \( Y \subseteq X^+ \) such that if

\[ y(x) = 0 \quad \forall y \in Y \text{ then } x = 0 \]

Def. \((X, \tau (X,Y))\) is the locally convex topology induced by the family of semi-norms \( \{ \gamma \} \) for each \( \gamma \in Y \)

\[ \gamma : X \to \mathbb{R}_+, \quad \gamma(x) = |y(x)| \quad \forall x \in X. \]
Equivalently, this is the weakest topology for which \( y : X \rightarrow Y \) is continuous \( \forall y \in Y \).

Remark 9. In this topology, \( U \) is a neighborhood of \( x_0 \in X \) iff \( \exists \varepsilon > 0 \) and \( \{y_1, \ldots, y_n \in Y \mid y \in Y \text{ such that } \} \)
\[ \{ x \in X \mid |y_i(x) - y_j(x)| < \varepsilon \text{ for } i = 1, \ldots, n \} \subseteq U \]

A sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq X \) is convergent to \( x \in X \) iff
\[ \lim_{n \to \infty} y(x_n) = y(x) \text{ for all } y \in Y. \]

Def. \( T(X, X^*) \) is called the weak topology on \( X \)
\( T(X^*, X^{**}) \) is called the weak topology on \( X^* \)
\( T(X^*, T(X)) \) is called the weak-* topology on \( X^* \)
and is usually denoted by \( T(X^*, X) \).

Remark 10. The weak topology on \( X \) (or \( X^* \)) is finer than the weak topology, in particular a sequence converging in the weak topology might not converge in norm while a sequence convergent in norm is always weakly convergent. Also a closed set in norm might not be weakly closed while all weakly closed sets are closed in norm. These two topologies coincide iff \( X \) (or \( X^* \)) is finite dimensional.

A sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq X^* \) is weak-* convergent to \( x^* \) iff it converges point-wise:
\[ \lim_{n \to \infty} x_n(x) = x^*(x) \text{ for } x \in X \]

The weak topology on \( X^* \) is finer than the weak-* topology. They coincide iff \( X \) is reflexive.
Theorem. If $(X, \| \cdot \|)$ is a normed space and $E \subseteq X$ is convex, then

$E$ is closed $(X, \| \cdot \|) \Rightarrow E$ is closed in $(X, \tau(X))$

The proof uses the geometric form of Hahn–Banach Theorem, i.e. any point and norm-closed convex set can be strictly separated by a hyperplane.

**Theorem (Hahn–Banach -- Bounded)** If $X$ is Banach and separable i.e. $\exists \{ x_1, x_2, \ldots, x_n, \ldots \} \subseteq X$ dense, then

$$B_{X^*} = \{ x^* \in X^* \mid \| x^* \|_* \leq 1 \}$$

is weak* sequentially compact, i.e.

$\forall \{ x^*_n \}_{n \in \mathbb{N}} \subseteq B_{X^*} \exists \{ x^*_n \}_{n \in \mathbb{N}} \subseteq \{ x^*_n \}_{n \in \mathbb{N}}$ and $x_0^* \in B_{X^*}$ such that

$$\lim_{n \to \infty} x^*_n(x) = x_0^*(x) \ \forall x \in X.$$

**Proof.** Hint: given $S = \{ x^*_n \subseteq B_{X^*} \}_{n \in \mathbb{N}}$ use Hahn–Banach (uniformly Lipschitz with $L = 1$) and for each $x \in X \{ x^*_n(x) \}_{n \in \mathbb{N}} \subseteq \mathbb{K}$ is bounded by $\| x \|_X$ hence relatively compact. Note that $X$ is not compact.

However, as in the proof of Arzelà–Ascoli Theorem, see Lecture 3, extract a subsequence $x^*_n \subseteq X$ convergent at each point in $\{ x_1, \ldots, x_n, \ldots \} \subseteq X$. Then, by density of $\{ x_1, \ldots, x_n, \ldots \} \subseteq X$ and equicontinuity of $\{ x^*_n \}_{n \in \mathbb{N}}$ the latter converges at each point in $X$.

Now show that the limit is in $B_{X^*}$. Note that:
Corollary If $X$ is Banach, $Y$ is normed and \( \{ T \in \mathcal{B}(X,Y) \mid T(x) = T(y) \exists x \neq y \in X \} \subseteq \mathcal{B}(X,Y) \), then \( T \in \mathcal{B}(X,Y) \). This follows from:

**Theorem (Uniform boundedness principle)** Let \( X, Y \) be as above then any \( S \subseteq \mathcal{B}(X,Y) \) is pointwise bounded i.e. for each \( x \in X \), sup \( \{ \| x^*(x) \| \mid x^* \in S \} \) is uniformly bounded, i.e. \( \exists \lambda > 0 : \| x^* \| \leq \lambda, \forall x \in S \).

**Remark 1** If $X$ is Banach any uniformly (pointwise) bounded set in \( X^* \) is relatively sequentially compact.

**Weaker compactness** (Extensions of Alaoglu-Banach):

**Theorem (Alaoglu-Banach)\)** If $X$ is a normed vector space then:

\[
B_{X^*} = \{ x^* \in X^* \mid \| x^* \| \leq 1 \}
\]

is compact in the weak $\ast$ topology.

**Theorem (Eberlein-Smulian)\)** If $X$ is Banach and \( E \subseteq X \), then \( E \) is weakly compact if and only if \( E \) is weakly sequentially compact.

**Remark 2** We can combine the two results to get: If $X$ is Banach and reflexive then:

1. \( B_{X^*} \) is compact and sequentially compact in the weak $\ast$ topology = weak $\ast$ topology.
2. \( B_X = \{ x \in X \mid \| x \| \leq 1 \} \) is weakly compact and sequentially compact.
Example 2 Let $D$ be open in $\mathbb{R}^n$ and $K = \mathbb{R}^d$. 

$$C_b(D, K) = \{ U: D \to K \mid U \text{ is continuous everywhere and bounded} \}$$

Then $\|U\|_b = \sup_{x \in D} |U(x)|$ defines a norm called the sup-norm. We have 

$$(C_b(D, K), \| \cdot \|_b)$$ 
is Banach.

$$(C_b(D, K), \| \cdot \|_b)$$ is separable $\implies D$ is bounded.

Example 3 Let $D \subseteq \mathbb{R}^n$ be open and $1 \leq p < \infty$.

$$C_p(D, K) = \{ U: D \to K \mid U \text{ continuous and } \int_D |U(x)|^p \, dx < \infty \}$$

Then $\|U\|_p = \left( \int_D |U(x)|^p \, dx \right)^{1/p}$ defines a norm on $C_p(D, K)$. We see that for $p \neq q$ 

$C_p \neq C_q$. In general for $D$ bounded, if 

$p < q$ then $C_p \subseteq C_q$.

However, none of these spaces are Banach.

Completing them:

$$C_p^\infty = C_p^\infty$$ 
is the $\| \cdot \|_\infty$ norm.

leads to spaces isomorphic to $L^p(D, K)$.

Example 4 Let $D \subseteq \mathbb{R}^n$ be open and $1 \leq p < \infty$.

$$L^p(D, K) = \{ U: D \to K \mid U \text{ measurable and } \int_D |U(x)|^p \, dx < \infty \}$$
Note that \( \| u \|_p = \left( \int |u(x)|^p \, dx \right)^{\frac{1}{p}} \) is not a norm on \( L^p(0,1) \) because

\[ \| u \|_p = 0 \iff u(x) = 0 \text{ almost everywhere} \]

which does not imply \( u = \text{constant function} 0 \) as needed for a norm on \( L^p(0,1) \).

But this issue can be fixed by realizing all functions that only differ on a set of measure 0 as a single element of the space \( L^0(0,1) \), or equivalently, by considering the localization of \( L^p(0,1) \) via the equivalence relation

\[ \| f \|_p = \| g \|_p \text{ almost everywhere}. \]

Theorem: If \( D \subset \mathbb{R}^n \) is open, then for \( 1 \leq p < \infty \)

\( (L^p(D, K), \| \cdot \|_p) \) are separable, Banach spaces and \( C_0(D, K) \) is a dense subspace.

However, \( L^0(D, K) \) cannot be obtained this way. Instead define for \( f : D \to K \), \( f \) measurable

\[ \text{ess sup} \, f = \inf \{ M > 0 \mid |f(x)| < M \text{ a.e.} \} \]

and

\[ L^0(D, K) = \{ f : D \to K \mid f \text{ measurable and } \text{ess sup} \, f < \infty \} \]

with norm (again functions differing only on a set of measure zero are considered one element)

\[ \| f \|_0 = \text{ess sup} \, f. \]
Theorem \( (L^\infty(O,K), \| \cdot \|_{\infty}) \) is a non-separable Banach space, and \( C_b(O,K) \subset L^\infty(O,K) \) is a closed subspace.

**Theorem (Interpolation in \( L^\alpha \) spaces)**

1. If \( \Omega \subset \mathbb{R}^n \) is an open set and \( f \in L^\alpha(\Omega, K), 1 \leq \alpha \leq \infty \), then
   \[ \| f \|_{L^\infty(\Omega, K)} \leq \| f \|_{L^\alpha(\Omega, K)} \] where \( \alpha \leq 1 \), and
   \[ \| f \|_{L^\alpha(\Omega, K)} \leq \| f \|_{L^\sigma(\Omega, K)} \] where \( \sigma \leq \alpha \)

2. If \( \Omega \subset \mathbb{R}^n \) is open, bounded, and \( f \in L^2(\Omega, K) \), \( 1 \leq \sigma \leq \infty \), then
   \[ f \in L^\infty(\Omega, K) \text{ for all } 1 \leq \sigma \leq 2 \] and
   \[ \| f \|_{L^\infty(\Omega, K)} \leq \| f \|_{L^2(\Omega, K)}^{\frac{2-\sigma}{\sigma}} \| f \|_{L^\sigma(\Omega, K)}^{\frac{\sigma-2}{\sigma}} \]

**Duals:** For \( 1 \leq \rho \leq \infty \)

\[ \left( L^\rho(O,K) \right)^* \cong L^{\rho'}(O,K) \text{ where } \frac{1}{\rho'} + \frac{1}{\rho} = 1 \]

\( \cong \) means isomorphic and isometric.

The isomorphism is given by

\[ x^* \in \left( L^\rho(O,K) \right)^* \stackrel{i^*}{\longrightarrow} f \in L^{\rho'}(O,K) \]

where \( x^* (u) = \int_{10} u(x, \nu) \, dx \) for \( u \in L^\rho(O,K) \)

Hence for \( 1 \leq \rho < \infty \)

\[ (L^\rho)^{**} \cong L^\rho \text{ with isomorphism} \]
\text{Let } u^* \in (L^p)^* \text{ such that for every } v^* \in (L^p)^* \text{ we have}
\[ u^*(v^*) = \int_0^1 u(x) v^*(v^*)(x) \, dx. \]

(The definition of \( v^* \) is above.) But we observe that \( v^* \) is in fact the inverse of the canonical map:
\[ f(v^*) = v^*(u) \]

\[ \Rightarrow \text{ } f \text{ surjective hence:} \]

\textbf{Theorem.} \( D \subseteq \mathbb{R}^n \text{ open and } 1 < p < \infty \)

\( L^p(D, \mathbb{K}) \text{ are reflexive Banach spaces.} \)

\textbf{Remark 13.} \((L^1)^* \neq L^\infty \) but \((L^\infty)^* \neq L^1 \)

So \( L^1 \) and \( L^\infty \) are not reflexive.

Similarly one can show that \( 1 < p < \infty \)
\[ (L^1)^* \neq L^\infty \]
\[ (C_p(D, \mathbb{R}))^* \neq L^1(D, \mathbb{R}) \]
\[ \frac{1}{p} + \frac{1}{q} = 1 \]

But this time none of \( C_p \) are reflexive.

\textbf{Watsonian.} From now on, using \( v^* \) we will identify the dual of \( L^p \) or \( C_p \), \( 1 < p < \infty \), with \( L^{p'} \) and write
\[ (L^p)^* = L^{p'} \quad ; \quad (C_p)^* = L^{p'} \quad 1 \leq p < \infty \]
\[ (L^\infty)^* = L^1 \]
Example 5 Solvabilty Spaces

Let $U \subseteq \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$, $1 \leq p < \infty$

$$W^{k,p}(U) = \left\{ f : U \to K \mid f \in L^p(U, K) \text{ and all its weak derivatives up to order } k \text{ are in } L^p(U, K) \right\}$$

with the norm:

$$\| f \|_{k,p} = \left( \sum_{|\alpha| \leq k} \| D^\alpha f \|_p^p \right)^{1/p}$$

is a Banach space. It is separable if and only if $1 \leq p < \infty$. It is reflexive if and only if $1 < p < \infty$.

**Def.** $\mathcal{R}$ has the cone property if there exists a finite cone $C$ such that for any $x \in \mathcal{R}$ there exists an element $c x$ congruent to $C$ with vertex $x$ and $c x \in \mathcal{R}$.

**Def.** (Strong local Lipschitz property)

$\mathcal{R}$ has the strong local Lipschitz property if for all $\varepsilon > 0$, a locally finite open cover $\{ U_j \}$ of $\mathcal{R}$ and a collection of $C^1$-functions such that

(i) $\bigcap U_j$ is open and

(ii) $\forall x, y \in U_j : \text{dist}(x, \partial U) < \varepsilon$ such that $|x - y| < \varepsilon$,

(iii) $|f_j(x) - f_j(y)| \leq M|x - y|$ for each $U_j$,
Theorem (embeddings in Sobolev spaces)

Assume $U \subset \mathbb{R}^n$ is open, has the cone property, $1 \leq \rho < \infty$ and $\nu \in \mathbb{N}$. Then

(A_{m,j}) If $\nu \rho < n$ and $j \in \{0,1,2,\ldots\}$

$$W^{m+\frac{j}{\rho}, \rho}(U) \subset W^{\rho, \rho}(U)$$

for $1 \leq \rho \leq n \rho/(n-\nu \rho)$

(B_{m,j}) If $\nu \rho = n$ and $j \in \{0,1,2,\ldots\}$

$$W^{m+\frac{j}{\rho}, \rho}(U) \subset W^{\rho, \rho}(U)$$

for $1 \leq \rho < \infty$

If in addition $\nu = 1$ and $m = n$, then

$$W^{m+\frac{j}{\rho}, \rho}(U) \subset C^0_{\text{Bd}}(\overline{U})$$

(C_{m,j}) If $\nu \rho > n$ and $j \in \{0,1,2,\ldots\}$

$$W^{m+\frac{j}{\rho}, \rho}(U) \subset C^0_{\text{Bd}}(\overline{U})$$

If in addition $\Lambda$ verifies the strong local Lipschitz property and $(\nu-1) \rho < n < \nu \rho$ then

$$W^{m+\frac{j}{\rho}, \rho}(U) \subset C^{0,\gamma}(\overline{U})$$

for $0 < \gamma \leq n-\nu \rho$

If in addition $(\nu-1) \rho = n$ then

$$W^{m+\frac{j}{\rho}, \rho}(U) \subset C^{0,\gamma}(\overline{U}), 0 < \gamma < 1$$

and if $\rho = 1$, $n = m+1$ then $\gamma = 1$ is allowed.
Theorem (Rellich - Kondrachov compactness)
Assume \( U \subset \mathbb{R}^n \) is open, bounded and has the cone property, \( 1 \leq p \leq \infty \) and \( n \geq 1, 2, \ldots, y \). Then

\( \text{(A}^c, j) \) If \( mp < n \) and \( j \in \{0, 1, 2, \ldots, y \} \)
\[ W^{m+\frac{j}{p}, p}(U) \xrightarrow{\text{compact}} W^{j, 2}(U) \]
for \( 1 \leq j < mp/(n-mp) \).

\( \text{(B}^c, j) \) If \( mp = n \) and \( j \in \{0, 1, 2, \ldots, y \} \)
\[ W^{m+\frac{j}{p}, p}(U) \xrightarrow{\text{compact}} W^{j, 2}(U) \]
for \( 1 \leq j < \infty \).

\( \text{(C}^c, j) \) If \( mp > n \) and \( j \in \{0, 1, 2, \ldots, y \} \)
\[ W^{m+\frac{j}{p}, p}(U) \xrightarrow{\text{compact}} C^{\beta}(\overline{U}) \]

If in addition \( A \) has the strong local dependence property then

\[ W^{m+\frac{j}{p}, p}(U) \xrightarrow{\text{compact}} C^{\beta}(\overline{U}) \]

and for \( (m-1)p < n < mp \)
\[ W^{m+\frac{j}{p}, p}(U) \xrightarrow{\text{compact}} C^{\beta, \gamma}(\overline{U}) \]
for \( 0 < \gamma < m - mp \).

Remark 14. These embeddings follow from the Sobolev inequality:

if \( 1 < p < \infty \) then \( \exists C = C(n, p) > 0 \) such that
In $\mathbb{R}^n$, \( u = w/(u-p) \) and any \( f \in C^1_c(\Omega^n) \):

\[
\| f \|_{L^2(\Omega^n)} \leq C \| f \|_{L^p(\Omega^n)},
\]

see for example Evans' book pages 263-265. The inequality is then combined with density arguments, for example:

\[
C^1_c(\Omega^n) \hookrightarrow \mathcal{W}^{1,1}_c(\Omega^n),
\]

and interpolation in \( L^p \) spaces, see Maddux 554 Lecture Notes 10, 11 for full proofs.

Sobolev spaces of functions that vanish on the boundary are defined via:

\[
C_c^\infty(\Omega, K) = \left\{ f : \Omega \to K \mid f \text{ has compact support strictly contained in } \Omega \text{ and is infinitely diff on } \Omega \right\}
\]

\[
W^{k,1}_0(\Omega, K) = C_c^\infty(\Omega, K) \text{ in } W^{k,1}(\Omega)
\]

Remark 15: \( W^{k,1}_0(\Omega, K) \) is in general strictly included in \( W^{k,1}(\Omega, K) \). Equality holds for \( K = \mathbb{R}^n \). The embeddings in Theorems above hold with \( W^{k,1}_0 \) replaced by \( W^{k,1} \), even without any assumptions on the boundary of \( \Omega \), \( \partial \Omega \).

Note that \( W^{k,2}_0 = H^k, k = 1,2, \ldots \)
Duality in Sobolev Spaces $1 \leq p < \infty$

Denote:

$$
(W_0^{k, p} (\Omega, K))^* = W^{-k, p'} (\Omega, K)
$$

This defines the Sobolev spaces for $k = -1, -2, \ldots$

Note that it implies:

$$
|| u ||_{-k, p} = \sup_{\Omega \neq 0} || u (v) ||
$$

and the generalized Holder inequality

$$
|| u ||_{-k, p} \leq || u ||_{k, p} || u ||_{-k, p}
$$

Important Note: Since in general we have the strict inclusion:

$$
W_0^{k+1, p} (\Omega, K) \subset W^{k+1, p} (\Omega, K)
$$

and $W_0^{k+1, p}$ is closed hence not dense in $W^{k+1, p}$.

We have:

$$
\forall \, u \in (W_0^{k, p})^* \quad \exists \, \psi \in (W_0^{k+1, p})^* \ni W_0^{k+1, p} \ni \psi
$$

But this is not an embedding as there are many $u$ for which the restrictions is the same.

**Remark 16** We have:

$$
(\star) \quad C_c^\infty (\Omega, K) = W^{-k, p'} (\Omega, K).
$$
Notation \( H^{-k, 2}(0, K) \)

Combining the embeddings of dual spaces, page 9, with embeddings for Sobolev spaces \( W^{k, p}, p > 0 \) we get:

If \( j = 0, 1, 2, \ldots \), and \( 1 \leq p' \leq \infty \) then for \( m > 0 \) with \( m_j < m \) we have

\[
W^{-j, 2} \hookrightarrow W^{-m-j, p'} \quad \text{dense}
\]

for all \( p' ; 1 \leq p' \leq \frac{m_j}{m-n+j} \)

While for \( m > 0 \) with \( m_j > m \) we have

\[
W^{-j, 2} \hookrightarrow W^{-m-j, p'} \quad \text{dense}
\]

for \( 1 \leq p' \leq \frac{m_j}{m-n+j} \)

where duality follows from (29) on the previous page!