4. Variational Methods

4.1. Euler-Lagrange Equations

4.2. An application to e-value problems and best constant in Poincaré inequality

4.3. Sufficient conditions for existence of minimizers: convexity and convexity

4.4. Another application to e-value problems - Rellich generalized compactness

4.5. Nonlinear e-value problems, best constant in Gagliardo-Nirenberg inequality, concentration compactness

4.6. Asymptotic functionals and orbital stability

4.1. Euler-Lagrange Equations

\[ X, \text{ Banach space over reals, } U \subseteq \mathbb{R} \text{ open} \]

\[ F : U \to \mathbb{R} \]

**Theorem (E-L without constraints)**

If \( u \in U \) is a local minima or maxima for \( F \) and the Gâteaux derivative of \( F \) at \( u \) in the direction \( \delta \in U \) exists then

\[ \frac{dF(u)}{\delta} = 0 \]
Proof: Assume \( v \in V \) is a local minimizer.
\[ \Rightarrow \exists \varepsilon > 0 \text{ such that} \]
\[ F(v + \varepsilon u) \geq F(v) \quad \forall \varepsilon \in \mathbb{R}, \ |\varepsilon| < \varepsilon_0 \]
\[ \Rightarrow \left\{ \begin{array}{l}
\frac{dF(v)}{du} = \lim_{\varepsilon \to 0} \frac{F(v + \varepsilon u) - F(v)}{\varepsilon} \geq 0 \\
\frac{dF(v)}{du} = \lim_{\varepsilon \to 0} \frac{F(v + \varepsilon u) - F(v)}{\varepsilon} \leq 0
\end{array} \right. \]
\[ \Rightarrow \frac{dF(v)}{du} = 0 \quad \square \]

**Theorem (Lagrange Multipliers):** Assume that

1° \( g : U \to \mathbb{R} \) is \( C^1 \);
2° \( A = \{ v \in U \mid g(v) = 0 \} \neq \emptyset ; \)
3° \( v \in A \) is a local minimizer for \( F|_A \);
4° \( \nabla g(v) \neq 0 ; \)
5° \( F \) is Fréchet differentiable at \( v \).

Then \( \exists \lambda \in \mathbb{R} \) such that

\[ \nabla F(v)[u] = \lambda \nabla g(v)[u] \quad \forall u \in X \]

**Proof.** Let \( v \in X \) such that \( \nabla g(v) \neq 0 \);

Define \( j : \mathbb{R}^2 \to \mathbb{R}^2 \quad \forall v \in X \)
\[ j(v, \theta) = (v + \theta \nabla v + \theta \nabla w) \]

Then \( j(0, 0) = g(v) = 0 \) since \( v \in A \)
\[ \frac{\partial j}{\partial \theta}(0, 0) = \nabla g(v)[u] \neq 0 \]
\[ = \text{by implicit function theorem} \exists \delta > 0 \text{ and} \]
\[ \mathcal{O} : (\mathbb{R}, 0) \rightarrow \mathbb{R} \] is a \( C^1 \) function such that
\[ f'_x(x, \theta(x)) = 0 \quad \forall x \in (\delta, \delta) \] hence
\[ u + \v + \Theta(x) w = u \quad \forall x \in (\delta, \delta) \]
Define \( i : (-\delta, \delta) \rightarrow \mathbb{R} \) by
\[ i(x) = f(u + \v + \Theta(x) w) \]
Then \( 0 \) is a local minimizer for \( i \). Since \( i \) is differentiable at \( 0 \) we have
\[ 0 = \frac{di}{dx}(0) = DF(u) [\v + \Theta'(0) x w ] \]
\[ = DF(u) [\v w ] + \Theta'(0) D F(u) [w ] \]
But from the implicit function theorem
\[ \Theta'(0) = - \frac{DG(u)[w]}{DG(u)[w]} \]
Hence
\[ DF(u)[w] = \frac{DF(u)[w]}{DG(u)[w]} DG(u)[w] \text{ for any } w \]
Let \( \Lambda = \frac{DF(u)[w]}{DG(u)[w]} \in \mathbb{R} \) and we are done!

Project: State and prove a similar theorem for a \( C^1 \) map:
\[ G : U \rightarrow Y \]
where \( Y \) is a Banach space!
4.2. An application to $l$-value problems and the best variance constant

Consider $\Omega \subset \mathbb{R}^n$ a bounded domain and

$$I: H_0^1(\Omega) \to \mathbb{R}$$

$$I[u] = \int_\Omega |\nabla u(x)|^2 \, dx$$

**Theorem.** If $u_0$ is a local minimizer of $I[u]$ subject to $\|u_0\|_{L^2(\Omega)} = 1$ then $u_0$ is a weak solution of:

$$-\Delta u_0 = \lambda u_0 \quad \text{for some } \lambda \in \mathbb{R}$$

**Proof.** Use Lagrange multiplier theorem:

$$\nabla I[u_0] \cdot \nabla \phi = \lambda \Delta (u_0) \cdot \nabla \phi \quad \forall \phi \in H_0^1$$

$$\Rightarrow \int_\Omega \nabla u_0 \cdot \nabla \phi \, dx = \lambda \int_\Omega u_0 \cdot \nabla \phi \, dx \quad \forall \phi \in H_0^1$$

but this is the def of weak soln's of $-\Delta u_0 = \lambda u_0$

**Theorem.** The problem:

$$\inf_{u \in H_0^1} I[u] = \lambda$$

$$\|u\|_{L^2} = 1$$

satisfies:

(i) It has a minimizer $u_0$, unique modulo multiplication by $e^{i\theta}$, $0 < \theta < 2\pi$.

(ii) $\lambda$ is the lowest $l$-value of $-\Delta$ on $\Omega$ with zero boundary condition.
(vii) If \( v_0 \) is an \( \varepsilon \)-function for \( \Omega \) then:

\[ v_0 \in C^2(\Omega) \cap C(\overline{\Omega}) \]

(iv) \( \lambda \) is a simple \( \varepsilon \)-value and the corresponding eigenfunction can be chosen strictly positive.

Proof: (i) Clearly \( 0 \leq \lambda < \infty \)

Let \( \{ u_k \} \subset H_0^1 \) be a minimizing sequence, i.e.

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \rightarrow \lambda \]

\[ \| u_k \|_{L^2} = 1. \]

Hence we have:

\[ \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \right) \| \nabla u_k \|_{L^2}^2 \rightarrow \lambda \]

\[ \Rightarrow \quad \| \nabla u_k \|_{L^2} \leq \sqrt{\lambda + 1} \quad \text{for } k \text{ sufficiently large} \]

\[ \Rightarrow \quad \| u_k \|_{H_0^1} = \| u_k \|_{L^2} + \| \nabla u_k \|_{L^2} = 1 + \| \nabla u_k \|_{L^2} \leq 2 + \sqrt{\lambda} \]

\[ \Rightarrow \quad \{ u_k \} \text{ bounded in } H_0^1 \quad \text{and} \quad H_0^1(\Omega), \text{ reflexive Banach space} \]

such that \( u_k \xrightarrow{k \to \infty} u_0 \) in \( H_0^1(\Omega) \)

which means \( v_0 \in \Omega \) in \( H_0^1(\Omega) \).
\[ \int \nabla u_k \cdot \nabla v \, dx \xrightarrow{\quad \text{as} \quad} \int \nabla u_0 \cdot \nabla v \, dx \]

On the other hand

\[ u_k \rightarrow u_0 \quad \text{in} \quad H^1_0(U) \]

\[ \implies \quad \nabla u_k \rightharpoonup \nabla u_0 \quad \text{in} \quad L^2(U) \]

\[ \text{Compactness} \]

\[ H^1_0(U) \hookrightarrow L^2(U) \]

Note that \( H^1_0(U) \hookrightarrow L^2(U) \) was discussed in Lecture 4, pages 21, 22, under the name Relllich - Kondratiev Theorem and relies on \( U \) bounded.

Now, the weak convergence implies:

\[ \int \nabla u_k \cdot \nabla v \, dx \xrightarrow{\quad \text{as} \quad} \int \nabla u_0 \cdot \nabla v \, dx \]

We claim:

\[ \int |\nabla u_0|^2 \, dx \leq \liminf_{\epsilon \rightarrow 0} \int |\nabla u_k|^2 \, dx = \lambda \]

Indeed, by Cauchy - Schwars inequality:

\[ \left( \int \nabla u_k \cdot \nabla u_0 \, dx \right) \leq \left( \int |\nabla u_k|^2 \, dx \right)^{\frac{1}{2}} \left( \int |\nabla u_0|^2 \, dx \right)^{\frac{1}{2}} \]

Apply \( \liminf \), note that \( \text{LHS is convergent} \).
\[
\left( \int |\nabla u_0|^2 \, dx \right)^2 \leq \int |\nabla u_0|^2 \, dx \cdot \inf_{u \in \mathcal{U}} \int |\nabla u|^2 \, dx.
\]

So \( I[u_0] \leq \lambda \), \( \lambda = I[u_0] \), \( \|u_0\|_{L^2} = 1 \Rightarrow \lambda \leq I[u_0] \) def. inf. minimum.

So \( u_0 \) is a minimizer, first part of (i) is done.

(ii) The fact that \( \lambda \) is an \( \epsilon \)-value with corresponding \( \epsilon \)-vector \( u_1 \) follows from previous theorem.

If \( \lambda_1 \) is another \( \epsilon \)-value with corresponding \( \epsilon \)-vector \( u_1 \), normalized to \( \|u_1\|_{L^2} = 1 \), we have

\[
\int \nabla u_1 \cdot \nabla u \, dx = \lambda_1 \int u_1 \cdot u \, dx \quad \forall \, \epsilon \in \mathbb{R}^N
\]

\[
\Rightarrow \quad I[u_1] = \lambda_1 \quad (\text{since } \|u| = u_1 \text{ above})
\]

\[
\|u_1\|_{L^2} = 1
\]

\[
\Rightarrow \quad \lambda \leq \lambda_1 \quad \text{since } \lambda = \inf_{\epsilon \in \mathcal{U}} I[\epsilon], \|\epsilon\|_{L^2} = 1
\]

(iii) Standard regularity theory for elliptic \( \epsilon \)-value problems.

For example, to obtain \( u_0 \in C^2(\Omega) \) consider \( \Omega_1 \subset \Omega \), i.e., \( \Omega_1 \) open and \( \Omega_1 \subset \text{interior of } \Omega \).

Let \( z_i \) be a \( C^\infty \) version of the characteristic function of \( \Omega_i \), i.e., \( z_i \in C^\infty(\mathbb{R}^N) \)

\[
\mathcal{K}_i(x) = 1 \quad \text{for } x \in \Omega_i, \quad \text{supp} \, z_i \subset \Omega_i
\]
Then $X_1 U_0$ satisfies in the weak sense

$$- \Delta (X_1 U_0) = \nabla (X_1 U_0) + \nabla (\nabla X_1 U_0)$$

on $\mathbb{R}^n$. Using Fourier Transform

$$|k|^2 \hat{X}_1 \hat{U}_0 (k) = \nabla \hat{X}_1 \hat{U}_0 (k) + \ldots \in L^2 (\mathbb{R}^n)$$

$$\Rightarrow (1 + |k|^2) \hat{X}_1 \hat{U}_0 (k) = (k+1) \hat{X}_1 \hat{U}_0 (k) + \ldots \in L^2 (\mathbb{R}^n)$$

$$\Rightarrow (1 + |k|^2) \hat{X}_1 \hat{U}_0 (k) \in L^2 (\mathbb{R}^n)$$

$$\Rightarrow \hat{X}_1 \hat{U}_0 \in \mathcal{H}^2 (\mathbb{R}^n) \text{ by Wiener's Th.}$$

$$\Rightarrow 2 \nabla X_1 \cdot \nabla \hat{U}_0 \in \mathcal{H}^2 (\mathbb{R}^n) \text{ and } X_1^2 \hat{U}_0 \in \mathcal{H}^2 \text{ and }$$

$$\Rightarrow - \Delta (X_1^2 U_0) = \nabla (X_1^2 U_0) + \nabla (2 \nabla X_1 \cdot \nabla U_0) \in \mathcal{H}^1$$

Repeat the argument above $\Rightarrow X_1^2 U_0 \in \mathcal{H}^3 (\mathbb{R}^n)$

Inductively $X_1^{m-1} U_0 \in \mathcal{H}^m (\mathbb{R}^n)$, hence by choosing $m$ large enough so that $(m-2) \cdot 2 > n$ we get

$$X_1^{m-1} U_0 \in C^2 (\mathbb{R}^n)$$

$$\Rightarrow U_0 \in C^2 (\mathbb{R}^n).$$

Since $\mathcal{R}_i \subset \mathcal{R}$ selections we get

$$U_0 \in C^2 (\mathbb{R}).$$
(iii) Since \(|\nabla |u_1| \leq |\nabla u_1| + u_1 H_0'(u_1)\)

we deduce that

\[ \tilde{u}_0 = |u_0| \in C^0(\Omega) \]

is also a minimizer.

Obviously \(\tilde{u}_0 \geq 0\) a.e. \(\quad \tilde{u}_0 \geq 0\)

By (iii) \(\tilde{u}_0 \in C^0(\Omega)\) on \(\Omega\).

If \(x_0 \in \Omega\) is such that \(\tilde{u}_0(x_0) = 0\) then

\(x_0 \in \mathcal{B}(x_0, \epsilon) \subseteq \Omega\) is a minimizer for \(\tilde{u}_0\) in

contradiction to the strong maximum principle.

\(\Rightarrow \tilde{u}_0 > 0\) on \(\Omega(\epsilon, x_0)\). \(\Rightarrow \tilde{u}_0 > 0\) on \(\Omega\).

So, we have that the minimizers can be chosen such that \(\tilde{u}_0 > 0\) on \(\Omega\).

For any other minimizer \(u_1\), we have that

\(\Rightarrow \Re u_1\), \(\Im u_1\) are eigenfunctions

Denote \(U_2 = \Re u_1\), \(U_3 = \Im u_1\)

Claim: \(\exists c_1, c_2 \in \mathbb{R}\) such that \(u_2 = c_1 U_2\), \(u_3 = c_2 U_3\)

Assume not, orthonormalize \(U_2\) and \(U_0\) and

renormalize \(U_2\) to have \(\|U_2\|_2 = 1\). Then

\[ \int_{\Omega} U_0 U_2 \, dx = 0 \Rightarrow U_2\] changes sign on \(\Omega\)

and since \(U_2\) remains an \(c\)-vector \(\Rightarrow |U_2|\) is

a minimizer, attaining its minimum on \(\Omega\)

in contradiction with the above argument.

In conclusion, \(U_1 = (c_1 + ic_2) U_0\), and hence

\[ \int_{\Omega} |u_1|^2 \, dx = 1 = \int_{\Omega} |u_0|^2 \, dx = c_1^2 + c_2^2 \leq 1. \]
This finishes part (i).

For any other eigenfunction $u_1$ for $\Lambda$, normalize it to $\|u_1\|_2 = 1$, which makes $u_1$ a minimizer for $I$, and so the above argument:

$$u_1 = c u_0 \text{ for some } c \in \mathbb{C}.$$ 

From the theory of compact operators we get that the invariant subspace of $-\Lambda$ corresponding to $\lambda$ is finite dimensional, and on this finite dimensional invariant subspace $\Lambda$ is given by a symmetric matrix. Since a symmetric matrix can be diagonalized by choosing bases of eigenvectors (eigenfunctions) via descent that the space is one dimensional since there is only one linearly independent eigenfunction.

So, the eigenvalue is simple and the eigenfunction can be chosen $u_0 \geq 0$. The theorem is now completely proven! \[ \square \]

Remark

$$\lambda = \inf_{u \in H_0^1} \int \frac{|\nabla u|^2}{u^2} \, dx = \inf_{u \in H^1_0} \frac{\int |\nabla u|^2 \, dx}{\int u^2 \, dx}$$

$$\|u\|_2^2 = 1 \quad \text{and} \quad u \geq 0$$

So the lowest $\lambda$-value of $-\Lambda$ on $C^2$ with

zero boundary condition is also the best constant for Painlevé inequality:

$$\|u\|_2^2 \leq C \|\nabla u\|_2^2 \quad \forall u \in H^1_0(\Omega)$$