Summary

1.2. Banach and Hilbert spaces (continuation)

Recall from Lecture 2 that the spaces

\[ C_p(D, \mathbb{R}) = \left\{ u : D \rightarrow \mathbb{R} \mid u \text{ continuous and} \right\} \]

\[ \int_D |u(x)|^p \, dx < \infty \quad \forall \quad 1 \leq p < \infty \]

with norm \( \|u\|_p = \left( \int_D |u(x)|^p \, dx \right)^{1/p} \) are not complete (not Banach spaces)! They can be completed by adding limits of fundamental (Cauchy) sequences in \( (C_p(D, \mathbb{R}), \|\cdot\|_p) \).

One arrives at the spaces \( L^p(D, \mathbb{R}) \) which have the equivalent characterization:

\[ L^p(D, \mathbb{R}) = \left\{ u : D \rightarrow \mathbb{R} \mid u \text{ measurable and} \right\} \]

\[ \int_D |u(x)|^p \, dx < \infty \quad \forall \quad 1 \leq p < \infty \]
Note that $\|u\|_p = \left(\int_D |u(x)|^p \, dx\right)^{1/p}$ is not a norm on $L^p(D, \mathbb{R})$ because

$$\|u\|_p = 0 \implies u(x) = 0 \text{ almost everywhere}$$

which does not imply $u = \text{constant function } 0$ = neutral element w.r.t. addition.

But this issue can be fixed by viewing all functions that only differ on a set of measure 0, as a single element of the space $L^0(D, \mathbb{R})$, or equivalently, by considering the factorization of $L^p(D, \mathbb{R})$ w.r.t. the equivalence relation

$$f \sim g \iff f = g \text{ almost everywhere}.$$

Theorem. If $D \subseteq \mathbb{R}^n$ is open then for $1 \leq p < \infty$

$(L^p(D, \mathbb{R}), \|\cdot\|_p)$ are Banach spaces,

and $C^0(D, \mathbb{R})$ is a dense subspace.

However $L^\infty(D, \mathbb{R})$ cannot be obtained this way.

Instead define for $f: D \to \mathbb{R}$ $\mathcal{M}$ measurable

$$\|\mu f\| = \inf \{M > 0 \mid |f(x)| < M \text{ a.e.}\}$$
and

\[ L^\infty(\mathbb{D}, \mathbb{R}) = \{ f : \mathbb{D} \to \mathbb{R} | f \text{ measurable and} \ \sup |f| < \infty \} \]

with norm (again functions differing only on a set of measure zero are considered one element)

\[ \|f\|_{\infty} = \sup |f| . \]

**Theorem** \((L^\infty(\mathbb{D}, \mathbb{R}), \|\cdot\|_{\infty})\) is a Banach space and \(C_0(\mathbb{D}, \mathbb{R}) \subset L^\infty(\mathbb{D}, \mathbb{R})\) is a closed subspace.

**Theorem** (interpolation in \(L^p\) spaces)

(i) If \( O \subseteq \mathbb{R}^n \) is an open set and \( f \in L^p(\mathbb{D}, L^q) \), \( 1 < p < \infty \) then \( f \in L^r(\mathbb{D}, \mathbb{R}) \) for all \( p < r < q \) and

\[ \|f\|_r \leq \|f\|_p^{1-a} \|f\|^q_{\infty} \text{ where } 0 < a < 1 \]

such that \( \frac{1}{r} = \frac{1-a}{p} + \frac{a}{q} \)

(ii) If \( O \subseteq \mathbb{R}^n \) is open, bounded, and
\[ f \in L^2, \ 1 < q \leq \infty \text{ then} \]

\[ f \in L^p(\mathbb{R}, \mathbb{R}) \text{ for all } 1 \leq p \leq q \text{ and} \]

\[ \|f\|_p \leq (\text{mes } \Omega)^{\frac{q-p}{q}} \|f\|_q \]

Sobolev Spaces

Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( k \in \mathbb{N}, 1 \leq p \leq \infty \)

\[ W^{k, p}(\Omega) = \left\{ f: \Omega \to \mathbb{R} \mid f \in L^p(\Omega, \mathbb{R}) \text{ and all its weak derivatives up to order } k \text{ are in } L^p(\Omega, \mathbb{R}) \right\} \]

with the norm:

\[ \|f\|_{k, p} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_p^p \right)^{\frac{1}{p}} \]

is a Banach space.
Theorem (embeddings in Sobolev spaces)
Assume \( \Omega \subset \mathbb{R}^n \) is open, has the cone property, \( 1 \leq p < \infty \) and \( m \in \{1, 2, \ldots, 5\} \). Then

(Am,i) If \( mp < n \) and \( j \in \{0, 1, 2, \ldots, 5\} \)

\[
W^{m+\frac{j}{p}, p} (\Omega) \hookrightarrow W^{j, q} (\Omega)
\]

for \( 1 \leq q \leq mp/(n-mp) \)

(Bm,i) If \( mp = n \) and \( j \in \{0, 1, 2, \ldots, 5\} \)

\[
W^{m+\frac{j}{p}, p} (\Omega) \hookrightarrow W^{j, \infty} (\Omega)
\]

for \( 1 \leq q < \infty \)

If in addition \( p = 1 \) and \( m = n \) then

\[
W^{m+\frac{j}{p}, 1} (\Omega) \hookrightarrow C^j_b (\Omega)
\]

(Cm,i) If \( mp > n \) and \( j \in \{0, 1, 2, \ldots, 5\} \)

\[
W^{m+\frac{j}{p}, p} (\Omega) \hookrightarrow C^j_b (\Omega)
\]

If in addition \( \Omega \) verifies the strong local Lipschitz property and \( (m-1)p < n < mp \) then
\[
\nu_{\omega + \delta}(\nu)(\Delta) \rightarrow C_{\delta}(k(\Delta)) \text{ for } \nu \geq 0
\]

If in addition \((\omega-1)\nu = \nu\) then

\[
\nu_{\omega + \delta}(\nu)(\Delta) \rightarrow C_{\delta}(k(\Delta)), 0 < \delta < 1
\]

and if \(\nu = 1\) then \(\gamma = 1\) is allowed.

Remark. \(\varphi \) has the core property if there exists a finite core \(C_x\) such that for every \(x \in \varphi \exists C_x
\]

\(\text{congruent to } C_1 \) with vertex \(x\) and \(C_x \subset \varphi\).

\(\varphi\) has the strong local Loxodromic property if \(\exists \gamma, \mu > 0\), a locally finite open cover \(U_j, j \in J\) of \(\varphi\) and \(f_j\) of \(\varphi\)-variables such that:

(i) \(\exists \gamma > 0\) such that every collection of \(K+1\) sets \(U_j\) has empty intersection.

(ii) \(\forall x, y \in U_j, \exists \gamma, \mu > 0\) such that \(d(x, y) < \gamma \), \(d(x, y) < \gamma \text{ and } d(x, y) < \gamma \text{ and } d(x, y) < \gamma \).

(iii) \(|f_j(z) - f_j(z)| < \mu, \forall z \in \mathbb{C}\).

(iv) In each \(U_j\) \(\exists \gamma > 0\) a cartesian coordinate system such that \(U_j \varphi_{\gamma}\) is given by \(\varphi_{\gamma} < f_j(x_1, \ldots, x_{n+1})\).
\[ C_c^\infty (U, \mathbb{R}) = \{ f : U \rightarrow \mathbb{R} \mid f \text{ has compact support strictly included in } U \text{ and is infinitely differentiable on } U \} \]

\[ \mathcal{W}_0^{k,1\sigma} (U, \mathbb{R}) = \overline{C_c^\infty (U, \mathbb{R})} \text{ in } \mathcal{W}^{k,1\sigma} (U, \mathbb{R}) \]

Remark: \( \mathcal{W}_0^{k,1\sigma} (U, \mathbb{R}) \) is in general strictly included in \( \mathcal{W}^{k,1\sigma} (U, \mathbb{R}) \). Equality holds for \( U = \mathbb{R}^n \). The embeddings in Theorem alone hold with \( \mathcal{W}^{k,1\sigma} \) replaced by \( \mathcal{W}_0^{k,1\sigma} \) even without any assumptions on the boundary of \( U \), \( \partial U \).

Notation: \( \mathcal{W}_0^{k,2} = H^k \), \( k = 1, 2, \ldots \)

Duality in normed spaces

Let \( X \) be a normed space (over \( K = \mathbb{R} \) or \( \mathbb{C} \)). Then

\[ X^* = \{ x^* : X \rightarrow K \mid x^* \text{ linear and continuous} \} \]

is a vector space over \( K \) called the dual of \( X \).
\[ \| x^* \| = \sup_{x \in X} \| x^*(x) \| \]

is a norm on \( X^* \) and \((X^*, \| \cdot \|)\) is a Banach space (even if \( X \) is not).

**Notation** \( X^{**} = (X^*)^* \)

There exists a canonical embedding:

\[ J : X \rightarrow X^{**} \]

\( J(x) \) evaluated at \( x^* \in X^* \) is \( x^*(x) \)

i.e.

\[ J(x)(x^*) = x^*(x), \quad \forall x^* \in X^* \]

\( J \) is linear, continuous and one-to-one (hence an embedding). Moreover:

\[ \| J(x) \|_{X^{**}} = \sup_{x^* \in X^*} | J(x)(x^*) | = \| x \| \]

\[ \| x^* \|_{X^{**}} \leq 1 \]
Remark Any normed space can be completed.

Indeed take

\[ \overline{J(X)} \subseteq X^{**} \text{ (w.r.t \; \| \cdot \|_w, \text{ weak})} \]

Since \( X^{**} \) is Banach, \( \overline{J(X)} \) will also be Banach!

Def (Reflexive Banach spaces) \( X \) is called reflexive iff \( J: X \rightarrow X^{**} \) is onto.

In this case \( J \) is an isomorphism and isometry between \( X \) and \( X^{**} \). Hence \( X \) is also Banach!

Examples For \( 1 \leq p < \infty \)

\[ \left( L^p(0,1;\mathbb{R}) \right)^* \cong L^{p'}(0,1;\mathbb{R}) \text{ where } \frac{1}{p} + \frac{1}{p'} = 1 \]

\( \cong \) means isomorphic and isometric.

The isomorphism is given by

\[ x^* \in (L^p(0,1;\mathbb{R}))^* \xrightarrow{\text{ip}} f \in L^{p'}(0,1;\mathbb{R}) \]

where

\[ x^*(u) = \int_0^1 f(x) u(x) \, dx \quad \forall u \in L^p(0,1;\mathbb{R}) \]
Here for $1 < p < \infty$

$$(L^p)^{**} \cong L^p \text{ with isomorphism}$$

$$u^{**} \in (L^p)^{**} \overset{i_p}{\longrightarrow} u \in L^p$$

where $u^{**}$ is such that for every $v^* \in (L^p)^*$ we have

$$u^{**}(v^*) = \int_0 u(x) i_p(v^*)(x) \, dx$$

(the definition of $i_p$ is above). But we observe that $i_p^*$ is in fact the inverse of the canonical map:

$$j(u)(v^*) = v^*(u)$$

$$\Rightarrow \ j \text{ is injective}$$

**Theorem**  \quad $\Omega \subset \mathbb{R}^n$ open and $1 < p < \infty$

$L^p(\Omega, \mathbb{R})$ are reflexive Banach spaces.
Rewrite: \((L^1)^* = L^\infty\) but \((L^\infty)^* \neq L^1\)

So \(L^1\) and \(L^\infty\) are not reflexive.

Similarly, one can show that for \(1 \leq p < \infty\)

\[
(C_p(D,\mathbb{R}^n))^* = L^{\frac{1}{1+\frac{1}{p}}}(D,\mathbb{R}^n)
\]

But this time none of \(C_p\) are reflexive.

Notation: From now on, using \(\mathcal{L}_p\) we will identify the dual of \(L^p\) or \(C_p\), \(1 \leq p < \infty\), with \(L^{p'}\) and write

\[
(L^p)^* = L^{p'}; (C_p)^* = L^{p'} \quad 1 \leq p < \infty
\]

Notation: Let \(U \subseteq \mathbb{R}^n\) open and \(1 \leq p < \infty\)

Denote:

\[
(W_0^{k,p}(D,\mathbb{R}^n))^* = W^{-k,p}(D,\mathbb{R}^n)
\]

This defines the Sobolev Spaces for \(k = -1, -2, \ldots\)

Note that it implies:
\[ \| u \|_{L^{2,\infty}} = \sup_{v \in \mathcal{C}_c^\infty(\mathbb{R}^n), \| v \|_{L^{2,\infty}} \leq 1} | u(v) | \]

and the generalized Hölder inequality
\[ | u(v) | \leq \| u \|_{L^{2,\infty}} \| v \|_{L^{2,\infty}} \]

It turns out that
\[ C_c^\infty(\mathbb{R}) \xrightarrow{\| \cdot \|_{L^{2,\infty}}} W^{-2,\infty}(\mathbb{R}, \mathbb{R}) \]

For \( p' = 2 \) we have the relation
\[ W^{-2,\infty}(\mathbb{R}, \mathbb{R}^n) \cong H^{-1,\infty}(\mathbb{R}, \mathbb{R}^n) \]

**Theorem.** If \( X, Y \) are normed spaces with
\[ X \xrightarrow{\text{dense}} Y \]

then \( Y^* \xrightarrow{\text{not necessarily dense}} X^* \)

Combining the above theorem with the embeddings for Sobolev spaces \( W_0^{k,p}, k \geq 0 \) we get:
If \( j = 0, 1, 2, \ldots \), and \( 1 \leq q' \leq \infty \), then for \( m > 0 \) with \( mg' < m \) we have

\[
W^{-8;2} \subseteq W^{-m-g',p'} \quad \text{for all } p' \text{ dense}
\]

while for \( m > 0 \) with \( mg' \geq m \) we have

\[
W^{-0;2} \subseteq W^{-m-g',p'} \quad \text{for } \frac{m}{2} \leq p' \text{ dense}
\]

Remark: Density follows from (26) see previous page.