Summary:

3.5. Nonlinear eq. (continued)
3.6. Applications to a nonlinear Schrödinger eq.

3.5 Nonlinear evolution equations (cont.)

\[
\begin{align*}
\frac{du}{dt} &= Au + f(t, u(t)) & t &> t_0 \\
&\left\{ v(t) \right. = v_0 \in X
\end{align*}
\]

(1)

\(X\) a Banach space \(A: D(A) \rightarrow X\) linear

\(f: (t_0, T) \times X \rightarrow X\). Recall:

Def (classical solution) \(U: \mathbb{R}^+ \rightarrow X\) continuous, continuously differentiable on \((t_0, t_1)\), \(U(t) \in D(A)\) for all \(t \in (t_0, t_1)\) and satisfying (1) is called a classical solution.

(Strong solution) \(U: \mathbb{R}^+ \rightarrow X\) continuous, \(du \in L^1((t_0, t_1), X)\), \(U(t) \in D(A)\) a.e. on \((t_0, t_1)\) and satisfying the eq in (1) a.e. and \(U(t_0) = v_0\) is called a strong solution.
If $A$ is the infinitesimal generator of a $C_0$ semigroup of bounded operators $T(t)$, $t \geq 0$, then any $U \in C([0, T], X)$ satisfying

$$U(t) = T(t - t_0)U_0 + \int_{t_0}^{t} T(t - s)f(s, U(s))\,ds$$

for $t_0 \leq t < t_1$ is called a mild solution.

**Lemma** If $A$ is the infinitesimal generator of a $C_0$ semigroup and $f: [t_0, T] \times X \to X$ is continuous then any classical or strong solution of (1) are also mild solutions.

**Theorem 1** If $A$ is the infinitesimal generator of a $C_0$ semigroup and $f: [t_0, T] \times X \to X$ satisfies

1. $f(\cdot, x)$ is continuous $\forall x \in X$
2. There exists $M > 0$ such that $\forall t \in [t_0, T], \forall x, y \in X$

$$\|f(t, x) - f(t, y)\| \leq M \|x - y\|$$

then (2) has a unique (mild) solution in $C([t_0, T], X)$.

**Theorem 2** If $A$ is the infinitesimal generator of a $C_0$ semigroup and $f: [0, T] \times X \to X$ satisfies (i) above and
(ii) \( \forall t, s, \epsilon \in [0, T], R > 0 \exists L(t, s, \epsilon, R) > 0 \)

such that for all \( t \in [t_0, t] \) and \( x, y \in X \) \( \| x \|, \| y \| \leq R \)

\[ \| f(t, x) - f(t, y) \| \leq L(t, s, \epsilon, R) \| x - y \| \]

then (2) has a unique maximal solution:

\[ U : [0, T_{max}) \rightarrow X \text{ continuous} \]

such that either \( T_{max} = T \) or \( \lim_{t \rightarrow T_{max}} U(t) = +\infty \)

Regular solutions of (1):

Theorem 3 Let \( A \) be the infinitesimal generator of a \( C_0 \)
semigroup \( T(t), t \geq 0 \), and \( (D(A), \| \cdot \|_A) \) be the \( \| \cdot \|_{\| \cdot \|_A} \)
Banach space with norm

\[ \| x \|_A = \| x \| + \| A x \| \]

Assume that \( f : [0, T] \times D(A) \rightarrow D(A) \) satisfies:

(i) \( f(\cdot, x) \) is continuous \( \forall x \in D(A) \)

(ii) \( \exists L > 0 \) such that \( \forall t \in [0, T], x, y \in D(A) \)

\[ \| f(t, x) - f(t, y) \|_A \leq L \| x - y \|_A \]

Then (1) has a unique classical solution on \([t_0, T] \) for \( v_0 \in D(A) \).
Proof: The proof of Theorem 1, see existence of mild solutions in lecture 14, page 11, continues through with \((X, || \cdot ||)\) replaced by \((V(4), || \cdot ||_4)\) and shows the existence of a unique \(C([t_0, T], V(4))\) solution of

\[ U(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s, u(s)) \, ds. \]

Let \(g(s) = f(s, u(s))\) clearly \(g: [t_0, T] \rightarrow V(4)\) is continuous (since \(f: [t_0, T] \times V(4) \rightarrow V(4)\) and \(u: [t_0, T] \rightarrow V(4)\) are both continuous). In particular,

\[ g(s) \in V(4) \quad \forall s \in [t_0, T] \]

and \( Ag(s) : [t_0, T] \rightarrow X \) is continuous.

(See Corollary 2, Lecture 14, page 7)

\[ U(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s, u(s)) \, ds \]

is a classical solution of:

\[
\begin{align*}
\frac{dv}{dt} &= Av + f(t, v(t)) \\
v(t_0) &= v_0
\end{align*}
\]

\( \Rightarrow U(t) = u(t) \) is a classical solution of (1).
Theorem 4 Let $A$ be the infinitesimal generator of a $C_0$ semigroup and $(D(A), \| \cdot \|)$ the Banach space with
\[
\| x \|_A = \| x \| + \| A x \|.
\]
Assume $f : [0, T) \times D(A) \to D(A)$ satisfies:
(i) $f(\cdot, x)$ is continuous $\forall x \in D(A)$
(ii) $\forall t_1, t_2 \in [0, T), R > 0 \exists L(t_1, t_2, R) > 0$:
\[
| f(t, x) - f(t, y) |_A \leq L(t_1, t_2, R) | x - y |_A
\]
for all $t \in [t_1, t_2]$ and $x, y \in D(A)$ with $\| x \|_A, \| y \|_A \leq R$.

Then for any $v_0 \in D(A)$ (i) has a unique classical
maximal solution
\[
U : (0, T_{\text{max}}) \to D(A)
\]
such that
\[
T_{\text{max}} = T \text{ or } \lim_{t \to T_{\text{max}}} U(t) = 0.
\]

Proof The proof of Theorem 2, see Lecture 14 page 13-19, continues even with $(X, \| \cdot \|)$ replaced by $(D(A), \| \cdot \|_A)$. The fact that the maximal (weld) solution is un
Just cloning follows as in Theorem 3 above.

Other ways to obtain regularity:

**Theorem 5.** Under the hypotheses of Theorem 1 if in addition \( f : [t_0, T] \times X \to X \) is continuously differentiable then the mild solution of (1) is a classical solution for all \( t_0 \in D(f) \).

**Sketch of proof:** Note that \( f \) continuously differentiable implies (i) and (ii) in Theorem 1, so we have a unique mild solution on \([t_0, T] \).

\[
U(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s, U(s)) \, ds
\]

For \( u_0 \in D(f) \), \( T(t-t_0)u_0 \) is continuously differentiable in \( t \), for \( t > t_0 \). To show that the integral term is differentiable in \( t \) one would like to move the \( t \) dependence to \( f \):

\[
\frac{d}{dt} \int_{t_0}^{t} T(t-s) f(s, U(s)) \, ds = \frac{d}{dt} \int_{0}^{t} T(t-s') f(s', u(s')) \, ds' \\
= T(t-t_0) f(t_0, U(t_0)) + \int_{0}^{t} T(t-s') \frac{\partial f}{\partial s}(s', u(s')) \, ds' \\
\text{ formally } + \int_{0}^{t} T(t-s') \frac{\partial f}{\partial u}(s', u(s')) \, ds' \\
= T(t-t_0) f(t_0, U(t_0)) + \int_{t_0}^{t} T(t-s) \frac{\partial f}{\partial s}(s, U(s)) \, ds \\
+ \int_{t_0}^{t} T(t-s) \frac{\partial f}{\partial u}(s, U(s)) \, ds \]
The problem is that we do not know whether \( \frac{du}{dt} \) even exists! To avoid this we use fixed point techniques to show that the equation for \( \frac{du}{dt} \) obtained by differentiating (2):

\[
(4) \quad \dot{W}(t) = T(t-t_0)A\dot{u}_0 + T(t-t_0)\int_{t_0}^t \beta(t, s, u(s)) \, ds + \int_{t_0}^t T(t-s) \frac{\partial \phi}{\partial u}(s, u(s)) \, ds
\]

continuously in \( t \) uniformly Lipschitz in \( u \).

has a unique solution. Then we show that \( W(t) \) is indeed \( \frac{du}{dt} \) by showing

\[
\lim_{n \to 0} \| u(t+h) - u(b) - W(t) \| = 0. \quad \text{This follows from using (2) and (x) to estimate}
\]

\[
W_n(t) = \frac{u(t+h) - u(t) - \chi(t)}{h}
\]

\[
\| W_n(t) \| \leq E(h) + M \int_0^t \| \chi_n(s) \| \, ds.
\]

(Where \( E(h) \to 0 \) as \( h \to 0 \) and \( M = \max \| T(t-s) \frac{\partial \phi}{\partial u}(s, u(s)) \| \).
By Gronwall \[ \| \psi_{n_0}(u) \| \leq \varepsilon \leq \psi_{n_0}(0) = 0 \]

hence \[ \psi_{n_0}(u) \rightarrow 0 \text{ as } u \rightarrow 0. \]

The last part, \( u(t) \in D(t) \forall t_0 \leq t < T \), and the fact that \( u \) satisfies (1) follows from

\[ u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s) f(s, u(s)) \, ds \]

is the classical solution of:

\[
\left\{ \begin{array}{l}
\frac{du}{dt} = Au + f(t, u(t)) \\
u(t_0) = u_0
\end{array} \right.
\]

see Corollary 2, Lecture 14, page 7. Since \( u = u \), \( u \) is a classical solution of (1). Q.E.D.

**Theorem 6.** Under the hypotheses of Theorem 1 of \( X \) is reflexive and \( f: [0, T] \times X \rightarrow X \) is Lipschitz (uniformly bounded) then the unique mild solution is a strong solution of (1) for all \( u_0 \in D(0) \).

**Sketch of proof.** From the formula for the mild solution (2) using that \( f \) is Lipschitz one gets:
\[ \| v(t + h) - v(t) \| \leq C h + M L \int h_0^t \| v(s + h) - v(s) \| ds \]

where \( M = \max \{ \| f(t) \| : b_0 \leq t \leq T \} \) and \( L \) is the Lipschitz constant for \( f \). By Gronwall:

\[ \| v(t + h) - v(t) \| \leq C h e^{ML(t-b_0)} \]

\[ \Rightarrow v \text{ is Lipschitz with constant } C e^{ML(t-b_0)} \]

\[ \Rightarrow S \mapsto f(s, v(s)) \text{ is Lipschitz} \]

\[ \Rightarrow v(t) = T(t-b_0) v_0 + \int b_0^t T(t-s) f(s, v(s)) ds \]

is the strong solution of

\[ \begin{cases} 
\frac{dv}{dt} = Av + f(t, v(t)) \\
v(b_0) = v_0 \in D(A) 
\end{cases} \]

See Corollary 3, Lecture 14 page 8. Then \( v(b) = v(t) \) is a strong solution of \((1)\). \( \Omega \in \mathbb{D} \).
3.5 Applications to a Nonlinear Schrödinger Eq

\begin{equation}
\begin{aligned}
\frac{\partial U}{\partial t} &= -i \nabla U + V(x)U + u|U|^2U \\
U(0) &= U_0(x)
\end{aligned}
\end{equation}

\text{\(U: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad u \in \mathbb{R}\)}

\text{\(V: \mathbb{R}^n \to \mathbb{R}, \quad V \in L^p(\mathbb{R}^n) \text{ for some } p > \frac{n}{2} \text{ and } p \geq 2.\)}

\text{\(K: \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \quad K \in L^\infty(\mathbb{R} \times \mathbb{R}^n)\)}

\text{Lemma 1: } \text{\(iH: H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) is the infinitesimal generator of a C\(^\infty\) group of unitary operators.}

\text{Proof: By Stone's Theorem, see Lecture 12, page 17, the conclusion is equivalent to}

\((-i)(iH) = \Delta \text{ is self-adjoint.}\)

\text{Recall: A symmetric operator } A: D(A) \to H \text{ with}

\text{a Hilbert space is self-adjoint if and only if}

\text{\(\text{Ran} \{ \pm iI \text{Id} - A \} = H.\)}

\text{In our case for } f \in L^2(\mathbb{R}^n) \text{ arbitrary}

\begin{equation}
(\pm i \text{Id} - \Delta) g = f
\end{equation}

\text{can be solved with a Fourier Transform:}
\[ (\pm i + |z|^{-2}) \hat{g}(z) = \hat{f}(z) \]

\[ \Rightarrow \hat{g}(z) = \frac{\hat{f}(z)}{|z|^2 + i} \]

\[ \Rightarrow g(x) = F.T^{-1} \left( \frac{\hat{f}(z)}{|z|^2 + i} \right) (x) \in H^2(\mathbb{R}^n) \]

Because \( \frac{\hat{f}(z)}{|z|^2 + i} \in \{ g \in L^2(\mathbb{R}^n) : (1 + |z|^2)g \in L^2(\mathbb{R}^n) \} \)

So (4) has solutions for all \( f \in L^2(\mathbb{R}^n) \)

\( \text{Rouge} (\pm \text{Id} - A) = L^2(\mathbb{R}^n) \Rightarrow A : H^2 \to L^2 \)

Self-adjoint \( \Rightarrow \) i\( A \) generates a C0 group of unitary operators.

\[ \text{Lemma 2} \]

\[ \|u\|_{\Delta} = \|u\|_{L^2} + \|Au\|_{L^2} \]

is an equivalent norm with the standard one on \( H^2(\mathbb{R}^n) \).

\[ \text{Proof. Recall:} \]

\[ \|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \sum_{i=1}^{n} \| \frac{\partial u}{\partial x_i} \|_{L^2}^2 + \sum_{i,j=1}^{n} \| \frac{\partial^2 u}{\partial x_i \partial x_j} \|_{L^2}^2 \]
\[
\text{But} \quad \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2}^2 = \left\langle \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_1} \right\rangle = \left\langle -i \frac{\partial}{\partial y}, \hat{u}(x), -i \frac{\partial}{\partial y}, \hat{u}(x) \right\rangle \\
= \left\langle \hat{u}(x), 14 |y|^2 \hat{u}(x) \right\rangle
\]

Hence
\[
\sum_{\delta=1}^{\infty} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2}^2 = \left\langle \hat{u}(x), 14 |y|^2 \hat{u}(x) \right\rangle
\]

Now
\[
\left\| \frac{\partial^2 u}{\partial x_1 \partial x_1} \right\|_{L^2}^2 = \left\langle \hat{u}(x), 14 |y|^2 \hat{u}(x) \right\rangle
\]

Hence
\[
\sum_{\delta=1}^{\infty} \left\| \frac{\partial^2 u}{\partial x_1 \partial x_1} \right\|_{L^2}^2 = \left\langle \hat{u}(x), 14 |y|^2 \hat{u}(x) \right\rangle
\]

\[
= \frac{1}{2} \left\| \hat{u}(x) \right\|_{L^2}^2 + \frac{1}{2} \left\| 14 \hat{u}(x) \right\|_{L^2}^2 \leq \left\| u \right\|_{H^2} \leq \left\| u \right\|_{L^2} + \left\| \Delta u \right\|_{L^2}
\]

Lemma 3
If $\forall(x) \in L^p(\mathbb{R}^n)$ with $p > n/2$ and $p > 2$ then for every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that
\[
\|u\|_{L^2} \leq C \|\Delta u\|_{L^2} + C(\epsilon) \|u\|_{L^2} \quad \forall u \in H^2(\mathbb{R}^n)
\]

Proof By Hölder
\[
(5) \quad \|u\|_{L^2} \leq \|u\|_{L^p} \|u\|_{L^{p'}} \quad \frac{1}{p} + \frac{1}{p'} = 1
\]

Since $p > \frac{n}{2}$, $n < \frac{2n}{n-4} = 4$, $\quad \|u\|_{L^2} \leq \|u\|_{L^{n-4}} \|u\|_{L^2} \quad \frac{1}{n-4} + \frac{1}{2} = \frac{1}{2}$
\[ \|u\|_{L^2} \leq C_n \| u \|_{H^2} \leq C_n \left( \|u\|_{L^2} + \| \Delta u \|_{L^2} \right) \]

for any \( u \in H^2 \). Let
\[ \overline{u}(x) = u(sx) \quad \text{for} \quad s > 0 \quad u \in H^2 \]

\[ = C_n \left( \frac{1}{s^{n/2}} \| u \|_{L^2} + \frac{s^2}{s^{n/2}} \| \Delta u \|_{L^2} \right) \]

\[ \|u\|_{L^2} \leq C_n s^{-n \left( \frac{1}{2} - \frac{1}{n} \right)} \|u\|_{L^2} + C_n s^{2 - n \left( \frac{1}{2} - \frac{1}{n} \right)} \|\Delta u\|_{L^2} \]

(5) implies \( \left( \frac{1}{2} - \frac{1}{n} = \frac{1}{p} \right) \)

\[ \|u\|_{L^2} \leq C_n \|u\|_{L^p} s^{2 - \frac{n}{p}} \|\Delta u\|_{L^2} + C_n \|u\|_{L^p} s^{-\frac{n}{p}} \|u\|_{L^2} \]

\( \forall \, \epsilon > 0 \quad \exists \, \delta > 0 \quad \text{such that} \quad \delta = C_n \|u\|_{L^p} s^{2 - \frac{n}{p}} \left( 2 - \frac{n}{p} > 0 \right) \)

and denoting \( C(\epsilon) = C_n \|u\|_{L^p} s^{-\frac{n}{p}} \) we get the conclusion.

\[ \text{Lemma 4} \quad i(A-V) : H^2 \rightarrow L^2 \quad \text{is the generator of a C0 group of unitary operators} \]

\[ \text{Proof} \quad \text{As in Lemma 1 (via Stone's theorem) is sufficient to prove that} \ A - V \text{ is self-adjoint} \]

Since \( A - V \) is symmetric (note \( V \) is real valued)
it suffices to prove \( \text{Range} \left( \pm i \text{Id} - \mathcal{A} + \mathcal{V} \right) = L^2 \)

or equivalently:

\[
\text{Range} \left( \text{Id} + i(\mathcal{A} - \mathcal{V}) \right) = L^2
\]

and

\[
\text{Range} \left( \text{Id} - i(\mathcal{A} - \mathcal{V}) \right) = L^2
\]

But \( i\mathcal{A}, -i\mathcal{V} \) are dissipative

\[
\text{Range} \left( \text{Id} - i\mathcal{A} \right) = L^2 \quad \text{(see Lemma 1)}
\]

\[
||-i\mathcal{V}u|| \leq \frac{1}{2} ||\text{Id}u|| + C||\mathcal{V}u||_{L^2} \quad \text{(see Lemma 3)}
\]

\[
\Rightarrow \quad \text{via perturbation of dissipative operators, see Lemma 6 in Lecture 13 page 9} = \)

\[
\text{Range} \left( \text{Id} - i\mathcal{A} + i\mathcal{V} \right) = L^2
\]

Some argument for \(-i\mathcal{A}\) and \(i\mathcal{V}\) gives

\[
\text{Range} \left( \text{Id} + i\mathcal{A} - i\mathcal{V} \right) = L^2 \quad \text{QED}
\]

**Lemma 5**

Let \( f : \mathbb{R}^2 \times H^2(\mathbb{R}^n) \rightarrow H^2(\mathbb{R}^n) \) \( n \geq 3 \),

\[
f(t, u)(x) = -i \mathcal{I} \nabla (t, x) u(x) \gamma^2 u(x), \quad R \in L^\infty(\mathbb{R} \times \mathbb{R}^n).
\]

Then \( f \) is well defined and \( \exists C > 0 \) such that

\[
||f(u) - f(v)||_{L^2} \leq C (||u||^2_{L^2} + ||v||^2_{L^2}) ||u - v||_{L^2}^2
\]
Proof
\[ f(u) - f(v) = -i k \left( |U|^2 u - |V|^2 v \right) \]
\[ = -i k \left[ (\bar{U}u + \bar{V}v)(u - v) + V^2 (\bar{U} - \bar{V}) \right] \]

So
\[ \| f(u) - f(v) \|_{L^2} \leq |k| \left( \| \bar{U}u + \bar{V}v \|_{L^2} \| u - v \|_{L^2} \right) \]
\[ + \| V^2 (\bar{U} - \bar{V}) \|_{L^2} \| u - v \|_{L^2} \]
\[ \leq 2 |k| \left( \| u \|_{L^\infty}^2 + \| V \|_{L^\infty}^2 \right) \| u - v \|_{L^2} \]

Since \( H^2(\Omega^m) \hookrightarrow L^\infty(\Omega^m) \) for \( n \leq 3 \) we obtain
\[ \| f(u) - f(v) \| \leq 2 |k| \mathcal{C} \left( \| u \|_{H^2}^2 + \| V \|_{H^2}^2 \right) \| u - v \|_{L^2} \]

Similarly we have:
\[ \| \Delta (f(u) - f(v)) \|_{L^2} \leq |k| \left( 2 \| u \|_{L^\infty}^2 + 2 \| V \|_{L^\infty}^2 \right) \| \Delta (u - v) \|_{L^2} \]
\[ + 2 |k| \left( \| \nabla (\bar{U}u + \bar{V}v) \|_{L^4}^4 + \| \nabla V^2 \|_{L^4} \right) \| \nabla (u - v) \|_{L^4} \]
\[ + |k| \left( \| \nabla (\bar{U}u + \bar{V}v) \|_{L^2}^2 + \| \Delta V^2 \|_{L^2} \right) \| u - v \|_{L^2} \]
which by $H^2(\Omega^n) \subset W^{1,4}(\mathbb{R}^n)$ for $n \leq 3$ implies

$$\| A (f(u) - f(v)) \|_{L^2} \leq C \left( \| u \|_{H^2}^{1/2} + \| v \|_{H^2}^{1/2} \right) \| u - v \|_{H^2}$$

Adding this to (x)

$$= \| f(u) - f(v) \|_{H^2} \leq \| f(u) - f(v) \|_{L^2} \leq C_1 \left( \| u \|_{H^2}^{1/2} + \| v \|_{H^2}^{1/2} \right) \cdot \| u - v \|_{H^2} \quad \text{QED}$$

**Theorem** If $n \leq 3$, $V$ is real valued $V \in L^p(\mathbb{R}^n)$ for some $p > n/2$ and $p > 2$, $U \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^n)$ and $V_0 \in H^2(\mathbb{R}^n)$ then (3) has a unique classical maximal solution:

$$U : (T_{min}, T_{max}) \to C$$

such that

$$T_{min} < 0 < T_{max}$$

$$T_{min} = -\infty \text{ on } \lim_{t \to T_{min}} \| U(t) \|_{H^2} = +\infty$$

$$T_{max} = +\infty \text{ on } \lim_{t \to T_{max}} \| U(t) \|_{H^2} = +\infty$$

**Proof** Lemmas 2, 4 and 5 imply the hypotheses of Theorem 4 at page 5 which implies the conclusion.