Summary

3.4 The abstract Cauchy problem (the core of unbounded operators)

3.5 Volterra evolution equations

3.4 The abstract Cauchy problem

Let $X$ be a Banach space and

$A: D(A) \rightarrow X$ a linear operator

(1) The homogeneous Cauchy problem:

$$\begin{cases}
\frac{du}{dt} = Au & t > 0 \\
\quad u(0) = x \in X
\end{cases}$$

Def $u: [0, T) \rightarrow X$ continuous, continuous differentiable on $(0, T)$ with $u(t) \in D(A)$ on $(0, T)$ satisfying (1) is called a classical solution of (1)
Theorem (uniqueness) Assume $D(A) = X$ and $R(n, A)$ exists for all $n > 0$ with

$$\sup_{n \to \infty} \| R(n, A) \| = 0$$

then (1) we have at most one classical solution for any $x \in X$.

Proof see Part Ch 4 Theorem 1.2.

Theorem (connection with semigroups of operators)

(a) If $A$ is the infinitesimal generator of a $C_0$ semigroup of bounded operators $T(t)$, $t \geq 0$ then (1) has a unique classical solution for all $x \in D(A)$ given by:

$$u(t) = T(t)x$$

(b) If (1) has a unique classical solution for all $x \in D(A)$ and all these solutions are continuously differentiable on $[0, T)$ for some $T > 0$ then $A$ is the infinitesimal generator of a $C_0$ semigroup of operators.
Proof (a) \( T(t)x, x \in D(A) \) is a classical solution follows from the properties of \( C_0 \) semigroups of operators. Uniqueness follows from previous theorem since \( A \) is infinitesimal generator of \( T(t) \) with \( \|T(t)\| \leq e^{\omega t} \) implies:

\[ R(\lambda, A) \text{ exists for } \lambda > \omega \] and

\[ \|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega} \]

\[ \Rightarrow \|R(\lambda, A)\|^{1/\lambda} \leq \frac{M^{1/\lambda}}{(\lambda - \omega)^{1/\lambda}} \rightarrow 1 \text{ as } \lambda \rightarrow \infty \]

\[ \Rightarrow \log \frac{\|R(\lambda, A)\|}{\lambda} \rightarrow 0 \text{ as } \lambda \rightarrow \infty \]

(b) See Part Chapter 4 Theorem 1.3.

Def If \( A \) is the infinitesimal generator of a \( C_0 \) semigroup of operators then

\[ U(t) = T(t)x, x \in X \setminus D(A) \] is called a mild solution of (1)
Remark: For any mild solution of (1) there exists a sequence of classical solutions $u_n(t)$ such that

$$u_n(t) \to u(t) \text{ uniformly on any } [0, T].$$

Indeed, for $x \in X \setminus Y(4)$ consider $x_n \in D(A)$ such that $x_n \to x$. Then

$$u_n(t) = T(t)x_n$$

are classical solutions and

$$\|T(t)x - T(t)x_n\| \leq M e^{wT} \|x - x_n\| \text{ on } 0 \leq t \leq T.$$

(ii) The nonhomogeneous Cauchy problem

Let $f : (0, T) \to X$

(2) \begin{align*}
\frac{du}{dt} &= Au + f(t) \quad 0 < t < T \\
U(0) &= x \in X
\end{align*}

Def 1° classical solution as before

2° Strong solution: $u : [0, T) \to X$ continuous

$$\frac{du}{dt} \in L^1([0, T), X), \quad U(t) \in D(A) \text{ a.e. on } (0, T) \text{ and }$$

$U$ satisfies (2) a.e. on $(0, T)$. 

3. Mild solution: If $A$ is the infinitesimal generator of a $C_0$ semigroup of operators $T(t), t \geq 0$ and $f \in L^1([0,T], X)$ then

$$U(t) = T(t)x + \int_0^T T(t-s) f(s) \, ds, \quad 0 \leq t \leq T$$

is called the mild solution of (2).

Remark: The mild solution is continuous on $[0,T]$.

**Theorem:** If $A$ is the infinitesimal generator of a $C_0$ semigroup of operators $T(t), t \geq 0$ and $f \in L^1([0,T], X)$ then any classical or strong solution of (2) is also a mild solution of (2).

**Proof:** Let $U(t)$ be a strong solution of (2). Fix $t > 0$ and denote:

$$g(s) = T(t-s)U(s)$$

then $g$ is differentiable a.e. on $0 < s < t$ and:

$$g'(s) = -T(t-s)AU(s) + T(t-s)U'(s) = \frac{T(t-s)f(s)}{L^1([0,T], X)}$$
Hence by integrating on \([0, t]\) we get

\[ y(t) - y(0) = \int_0^t T(t-s)f(s)\,ds \quad \text{or} \]

\[ U(t) - TU(t)X = \int_0^t T(t-s)f(s)\,ds \quad \text{QED} . \]

**Reciprocals:** Let \( V(t) = \int_0^t T(t-s)f(s)\,ds \)

**Theorem 2** If \( A \) is infinitesimal generator of a \( C_0 \) semigroup \( T \in L^1([0,T), X) \) and \( f \) continuous on \((0, T)\) then

(2) \( u \) is a classical solution on \([0, T) \forall x \in D(A) \) if

(i) \( u(t) \) is continuously differentiable on \((0, T)\),

or equivalently,

(ii) \( u(t) \in D(A) \forall 0 < t \leq T \) and \( Au(t) \) is continuous on \((0, T)\).

**Proof** Classical solutions \( \Rightarrow \) (i)

\[ U(t) = U(0) + \int_0^t U(t-s)A\,ds \]

\[ \Rightarrow \quad V(t) = U(t) - T(t)X = \int_0^t T(t-s)f(s)\,ds \quad \text{continuously differentiable on } (0, T) \forall X \in D(A) \]

(i) \( \Rightarrow \) (ii) For \( h > 0 \) and \( 0 < t \leq T \) we have:

\[ (3) \quad \frac{T(t+h)-T(t)}{h} u(t) = \frac{V(t+h) - V(t)}{h} = \frac{1}{h} \int_t^{t+h} T(t+s)f(s)\,ds \]
From $f$ continuous on $[β, b + ϵ]$ we have RHS less than limit value $\lim_{κ→0} u(κ) = u(b) ∈ L^1(0,T)$ and

$$Au(b) = u'(b) - f(b)$$

$$= \to Au(b) \text{ continuous on } (0,T)$$

(ii) $⇒$ classical solutions

Now $L + IS$ in (3) and integral term in RHS have limits as $κ → 0$ we have

$$Au(b) = u^+v(b) - f(b) \text{ or } u^+v(b) = Au(b) + f(b)$$

hence $u^+v(b)$ is continuous on $(0,T) ⇒ u^+$ exists on $(0,T)$ is continuous and

$$u'(b) = Au(b) + f(b)$$

Hence $v(b) = T(b)x + V(b)$ is continuously differentiable on $(0,T)$ for $x ∈ D(\epsilon)$ and

$$u'(b) = AT(b)x + Au(b) + f(b) = Au(b) + f(b)$$

Therefore $v(b) = T(b)x + V(b)$ is a classical solution (QED)

Corollary 2.1: A $\sigma$-finite $\text{differential}$ generator of a $C_0$-semigroup $T(\epsilon, T), x]$, $f$ continuous on $(0,T)$ and

(i) $f$ is continuously differentiable on $[0,T]$; or

(ii) $f(κ) ∈ D(α)$ for $0 < κ < T$ and $A\{f(κ) ∈ L^1([0,T), x]$; then (2) has a unique classical solution for every $x ∈ D(α)$. 
Proof
(i) \( v(t) = \int_0^t T(t-s) f(s) \, ds = \int_0^t T(s) f'(t-s) \, ds \)

\( \Rightarrow v(t) = T(t) f(0) + \int_0^t T(s) f'(t-s) \, ds. \)

\( \Rightarrow v \) continuously differentiable on \((0, T)\)

(ii) By \( A \) closed:
\[
A v(t) = A \int_0^t T(t-s) f(s) \, ds = \int_0^t A T(t-s) f(s) \, ds
\]
\[
= \int_0^t T(t-s) A f(s) \, ds \quad \text{continuous on } (0, T)
\]

Remark. If \( v \) is the mild solution of (2) then \( v \) is the uniform limit on \([0, T]\) of classical solutions of (2).

Indeed, let \( \{f_n\} \subseteq C^4[0, T], X \) such that
\[
\lim_{n \to \infty} \int_0^T \| f_n(t) - f(t) \| \, dt \to 0 \text{ as } n \to \infty.
\]
and \( \{x_n\} \subseteq B(\mathbb{R}) \) such that \( x_n \to x \). Then
\[
\left\{ \begin{array}{l}
\frac{d u_n}{dt} = A u_n + f_n(t) \\
u_n(0) = x_n
\end{array}\right.
\]
\(\text{has classical solutions } u_n(t) = T(t) x_n + \int_0^t T(t-s) f_n(s) \, ds\)
and \( u_n \to u \) on \([0, T] \)

**Theorem 3** If \( A \) is a infinitesimal generator of a \( C_0 \) semigroup \( \{S(t)\}_{t \geq 0} \) and \( f \in L^1([0, T], X) \) then (2) has a strong solution \( u \) on \([0, T] \), \( \forall x \in D(A) \) if

(i) \( u(t) \) is differentiable a.e. on \((0, T)\) and \( u \in L^1([0, T], X) \),

or equivalently:

(ii) \( u(t) \in D(A) \) a.e. on \((0, T)\) and \( Au(t) \in L^1([0, T], X) \).

**Proof** Adapt the one for Theorem 2.

**Corollary 3** If \( f \in L^1([0, T], X) \) or \( f \) is Lipschitz continuous on \([0, T] \) and \( X \) is reflexive, then (2) has a unique strong solution for any \( x \in D(A) \).

### 3.5 Nonlinear evolution equations

Let \( X \) be a Banach space:

\[
\begin{cases}
\frac{du}{dt} = Au + f(t, u(t)) & t > 0 \\
u(t_0) = u_0 \in X
\end{cases}
\]

where \( A : D(A) \to X \) is the infinitesimal generator
of a $C_0$ semigroup of operators and $f: \mathbb{R}_0 \times X \to X$.

Lemma. If $U$ is a classical or strong solution of (4) and $f: \mathbb{R}_0 \times X \to X$ is continuous then $U$ satisfies:

\[(5) \quad U(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s, U(s)) \, ds\]

Proof. As before, fix $t > t_0$ and denote

\[g(s) = T(t-s)U(s)\]

Then $g$ is differentiable a.e. on $t_0 < s < t$ and

\[g'(s) = -T(t-s)AU(s) + T(t-s)f'(s)\]

\[= T(t-s)f(s, U(s))\]

continuous on $[t_0, t]$.

By integration,

\[g(t) - g(t_0) = \int_{t_0}^{t} T(t-s)f(s, U(s)) \, ds.\]

\[\Rightarrow U(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s, U(s)) \, ds \quad \text{Q.E.D.}\]
Def. A continuous selection of (5) is called a mild solution of (4).

Theorem (existence of mild solutions) If \( A \) is the infinitesimal generator of a \( C_0 \) semigroup of operators and \( f : [t_0, t] \times X \to X \) is continuous in \( t \) and

\[
\| f(t, u) - f(t, v) \| \leq L \| u - v \| \quad \forall t \in [t_0, T], \quad u, v \in X
\]

then problem (4) has a unique mild solution that depends continuously on the initial data \( u_0 \).

Proof. Consider

\[
K : C([t_0, T] \times X, X) \to C([t_0, T] \times X, X)
\]

\[
(Ku)(t) = \int_{t_0}^{t} T(t-s) f(s, u(s)) \, ds
\]

Continuity of \( f \) \( \Rightarrow \) \( K \) well-defined.

\( f \) Lipschitz \( \Rightarrow \) \( K \) Lipschitz.

To make \( K \) contractive use Bielecki norms:

\[
\| u \|_{\delta} = \sup_{t_0 \leq t \leq T} e^{-\delta(t-t_0)} \| u(t) \| \quad \text{with} \quad \delta > \nu + ML
\]
$$\| (Ku_1 - Ku_2)(t) \| \leq \int_{t_0}^{t} \| T(t-s)\| \| f(s, u_1(s)) - f(s, u_2(s)) \| ds$$

$$\leq \int_{t_0}^{t} m e^{\omega(t-s)} \| u_1(s) - u_2(s) \| ds$$

$$\leq \| u_1 - u_2 \| \int_{t_0}^{t} m e^{\omega(t-s)} e^{\delta(s-t_0)} ds$$

$$= mL \frac{e^{\omega t} e^{(\delta-\omega) s}}{\delta-\omega} \left| \frac{t}{t_0} - e^{-\delta t_0} \right| \| u_1 - u_2 \| \sigma$$

$$\Rightarrow \sup_{t_0 \leq s \leq t} e^{-\delta(\delta - t_0)} \| (Ku_1 - Ku_2)(t) \|$$

$$\leq mL \frac{(1 - e^{(\delta-\omega)(t-t_0)})}{\delta-\omega} \| u_1 - u_2 \| \sigma$$

$$\Rightarrow \| Ku_1 - Ku_2 \| \sigma \leq \frac{mL}{\delta-\omega} \| u_1 - u_2 \| \sigma$$

with $mL / (\delta-\omega) < 1$.

$$\Rightarrow u(0) = T(0-t_0) u_0 + (Ku)(0)$$ has a unique solution in $C([t_0, T], X)$ for any $u_0 \in X$.

Moreover, if $U$ solves $U(t) = T(t-t_0) u_0 + Ku(t)$ we have

$$\| U - \hat{U} \| \sigma \leq mL \| u_0 - \hat{u}_0 \| + \frac{mL}{\delta-\omega} \| U - \hat{U} \| \sigma$$

$$\Rightarrow \| U - \hat{U} \| \sigma \leq \frac{mL}{1 - (mL)/\delta-\omega} \| u_0 - \hat{u}_0 \|$$
hence \( \sup_{0 \leq t \leq T} \| u(t) - v(t) \| \leq \frac{Me^{\int_{t-\theta}^{T} f(s) \, ds}}{1 - \frac{\omega}{e}} \| u_0 - v_0 \| \)

\( \Rightarrow \) the mild solution depends continuously \((C^0)\) on the initial data. \( \square \)

**Theorem (local existence and maximal solutions)**

Let \( A \) be the infinitesimal generator of a \( C_0 \) semigroup \( T(t), t \geq 0 \). Assume \( f : [0, T) \times X \to X \) satisfies:

(a) \( f(\cdot, x) : [0, T) \to X \) is continuous \( \forall x \in X \)

(b) \( \forall \, t_0, t_1 \in [0, T), \forall \mu > 0 \exists \, L(t_0, t_1, \mu) \)

such that for all \( t \in [t_0, t_1] \) and \( u, v \in X \) with \( \| u_0 \|, \| v_0 \| \leq M \),

\[ \| f(t, u) - f(t, v) \| \leq L(t_0, t_1, \mu) \| u - v \| \]

then for any \( x \in X \) the problem

\[
\begin{align*}
\frac{dU}{dt} &= AU + f(t, U) & 0 < t < T \\
U(0) &= x
\end{align*}
\]

has a unique mild solution \( U : [0, T_{\max}) \to X \) such that either

(i) \( T_{\max} = T \)

or (ii) \( \lim_{t \to T_{\max}} \| U(t) \| = +\infty \)

\( t \to T_{\max} \)
Proof. First we show "local existence":

\[
\forall t_0 \in [0,T), \forall x \in X \exists T(\tau_0, ||u_0||) \subset (t_0, T) \text{ such that }
\]

\[
(5) \quad u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s, u(s)) \, ds
\]

has a unique solution in \( C([t_0, T(\tau_0,||u_0||)], X) \).

Recall that \(|T(\tau)| \leq M e^{\omega \tau}\). Choose \( \delta > 0 \) such that

\[
e^{\omega \delta} \leq 2
\]

For \( t_0 \in [0,T) \), \( u_0 \in X \) and let \( t_i = \min \{ t_0 + \delta, \frac{t_0 + \delta}{2} \} \)

\[
K : C([t_0, t_i], X) \to C([t_0, t_i], X)
\]

\[
(Ku)(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s, u(s)) \, ds
\]

Now we seek \( t_0 < T_i \leq t_i \) such that:

\[
K : C([t_0, T_i], B(0, R)) \to C([t_0, T_i], B(0, R))
\]

\[
R = 4M ||u_0||
\]

where \( B(0, R) = \{ x \in X | ||x|| \leq R \} \).

In other words, \( K \) leaves the ball of radius \( R \) invariant in the Banach space \( C([t_0, T_i], X) \) endowed with supremum norm.
We use that for $b \leq s \leq t$: 

$$\| U(t-s) U \| \leq \rho e^{-\omega(t-s)} \leq \rho e^{-\omega t} \leq 2M$$

$$\| f(s, v) - f(s, 0) \| \leq L(t_1, t_2, R) R \text{ for } v \in B(0, R)$$

to get:

$$\| (Kv)(t) \| \leq 2M \| v \| + \int_0^t 2M L R ds$$

$$+ \int_0^t 2M \| f(s, 0) \| ds$$

$$\leq 2M \| v \| + 4M \| v \| + 2M L (t-t_0) + 2M F(t-t_0)$$

where $F = \sup_{b \leq s \leq t} \| f(s, 0) \|$

To have $\| (Kv)(t) \| \leq R = 4M \| v \|$ for $b \leq t \leq T$, it suffices to choose $b \leq T_1 \leq t$, such that

$$2M \| v \| + 4M \| v \| + 2M L (t-t_0) + 2M F(t-t_0) \leq 4M \| v \|$$

Hence

$$T_1 = \min \{ t_1, t_0 + \frac{2M \| v \|}{8M L \| v \| + 2M F} \}$$

With this choice

$$L: C([b_0, T], B(0, R)) \to C([b_0, T], B(0, R))$$

is well defined. Moreover, for $b \leq t \leq T_1$:
\[ \left\| \langle u_1 - u_2 \rangle (t) \right\| \leq \int_{t_0}^{t} 2 \langle 1 - \langle v_1 (s) - v_2 (s) \rangle \rangle \, ds \\
\leq \frac{4 M^2}{8 M^2 - 1} \left\| u_0 \right\| \sup_{t_0 \leq s \leq t_1} \left\| (v_1 (s) - v_2 (s)) \right\| \\
= \frac{1}{2} \sup_{t_0 \leq s \leq t_1} \left\| (v_1 (s) - v_2 (s)) \right\| \\
Hence, \ K \ is \ a \ contraction \ with \ Lipschitz \ constant \ \frac{1}{2} \ in \ the \ complete \ metric \ space \ \mathcal{C}([t_0, T_1], \mathbb{R}) \ \text{endowed with the supremum metric}: \\
d_{\infty} (u, \tilde{u}) = \sup_{t_0 \leq s \leq t_1} \left\| u(s) - \tilde{u}(s) \right\| \\
(\text{Note that } \mathbb{B}(0, R) = \{ x \in \mathbb{R} \mid \left\| x \right\| \leq R \} \ is \ closed). \\
Consequently, \ K \ has \ a \ unique \ fixed \ point \ which \ is \ equivalent \ to \ (5) \ has \ a \ unique \ solution: \\
U \in \mathcal{C}([t_0, T_1], \mathbb{B}(0, R)). \\
Consider \ another \ solution \ of \ (5) \\
\bar{v} \in \mathcal{C}([t_0, T_1], \mathbb{R}) \\
and \ let \\
\bar{Y} = \{ t \in [t_0, T_1] \mid \bar{v}(s) = \bar{v}(s) \ for \ t_0 \leq s \leq t \} \\
then \ t_0 \in \bar{Y} \ since \ \bar{v}(t_0) = \bar{v}(t_0) = v_0. \\
\bar{Y} \ is \ closed \ since \ both \ \bar{v}, \ \bar{Y} \ are \ continuous
If $t_2 \in J$, $t_2 < T_1$, then from the choice of $T_1$ we get $\|U(t_2)\| < R \implies 
abla U(t_2) = 0$ and from continuity of $\bar{U}$ there exists $T_2 > t_2$ such that $\|U(t)\| < R$ on $[b_0, t_2] \leq T_2$.

Since $V \subset C([b_0, t_2], \mathcal{B}(X))$ is contractive we get $U(t) = \bar{U}(t)$ on $[b_0, T_2]$ so $[b_0, t_2 + T_2 - t_2) \leq J = \emptyset$ is open.

In conclusion, $U(t) = \bar{U}(t)$ on $[b_0, T_1]$ and (5) has a unique solution in $C([b_0, T_1], X)$.

**Maximal solution**: Consider

\[(6) \quad U(t) = T(t)\varphi + \int_0^t T(t-s) f(s, U(s)) \, ds \quad 0 \leq t \leq T_1\]

Let $S = \{ U : [0, T] \to X \mid U \text{ continuous, } U \text{ satisfies (6) on } [0, T] \}$.

$S + \varphi$: Use the local existence for $t_0 = 0$ and choose $T = T(0, \|x_1\|)$.

Define the order relation on $S$

$U_1 \leq U_2$ if $U_2$ is defined on a larger interval than $U_1$ and $U_1(t) = U_2(t)$ on their common interval of definition.
Consider a totally ordered subset \( C \subset S \) indexed by \( j \).

\[
C = \{ U_j : I_j \to X \mid j \in J \}, \quad U_{j_1} \leq U_{j_2} \text{ or } U_{j_2} \leq U_{j_1}, \forall j_1, j_2 \in J
\]

then define

\[
U_C : \bigcup_{j \in J} U_I \to X
\]

by \( U_C(t) = U_j(t) \) if \( t \in I_j \).

Note \( U_C \) is well defined: for \( t \in I_j \cap I_{j_2} \) case have \( U_{j_2} \leq U_j \) or \( U_j \leq U_{j_2} \), in any case \( U_j(t) = U_{j_2}(t) \).

\( U_C \) is continuous, since for \( t \in I_j \), \( \exists \varepsilon > 0 \) such that \( (t - \varepsilon, t + \varepsilon) \subset I_j \Rightarrow U_C = U_j \) on \( (t - \varepsilon, t + \varepsilon) \), \( U_C \) continuous.

\( U_j \leq U_C \forall j \in J \).

So \( U_C \) is an upper bound for \( C \). By Zorn's Lemma, \( S \) has a maximal element

\[
U : [0, T_{\max}) \to X \text{ continuous, } U \text{ satisfies (6) on } [0, T_{\max}). \text{ Hence } U \text{ is a mild solution of the problem}
\]

if \( U : [0, T_{\max}) \to X \) is another maximal element, then for \( \forall j \in J \) consider \( T_{\max} \leq T_{\max} \) and

\[
J = \{ t \in [0, T_{\max}) : U(s) = \overline{U}(s) \text{ for } 0 \leq s \leq t \}
\]

then \( 0 \in J \), \( J \) is closed by continuity of \( U \) and \( \overline{U} \), and \( J \) is open.
since for $t_0 \in T \exists \exists T(t_0, \|u(t_0)\|) \subseteq [t_0, T_{max})$

such that (5) has a unique solution in $C([t_0, T(t_0, \|u(t_0)\|)], X)$ but both $u$ and $\tilde{u}$ are solutions of (5) in $C([t_0, T(t_0, \|u(t_0)\|)], X)$ hence

$[0, T(t_0, \|u(t_0)\|)] \subseteq \tilde{Y} \Rightarrow \tilde{Y}$ open.

So $\tilde{Y} \subseteq \tilde{Y} \cap [0, T_{max}) \supseteq \tilde{Y} \subseteq \tilde{Y}$.

Finally if $T_{max} < T$ and line $\|u(t)\| < \infty$

then consider $\tilde{T} < T_{max}$ and

$Y = \sup \{\|u(t_n)\| \mid t_n \in \tilde{T} \}$

Note that

$(f_n) u(t) = T(t - t_n) u(t_n) + \int_{t_n}^{t} T(t - s) f(s, u(s)) \, ds$

But by local existence $(f_n) u$ has a unique solution in $C([t_n, T(t_n, \|u(t_n)\|)], X)$ and

$T(t_n, \|u(t_n)\|) - t_n \geq T(t_1, N) - t_1 > 0$

$\Rightarrow U$ can be extended on $[0, T_{max} + T(t_1, N) - t_1)$ in contradiction with the maximality of $U$. 