Summary

- 3.3 Perturbations of infinitesimal generators
- 3.4 Abstract Cauchy Problem

3.3. Perturbations of infinitesimal generators

**Theorem (perturbation by bounded operators)**

Let \( X \) be a Banach space, \( A \) an infinitesimal generator of a \( C_0 \) semigroup \( T(\cdot) \), \( \| T(t) \| \leq M e^{\omega t}, t \geq 0 \).

If \( B : X \to X \) is linear and bounded, then \( A + B \) is the infinitesimal generator of a \( C_0 \) semigroup \( S(\cdot) \), \( \| S(t) \| \leq M e^{(\omega + \| B \|) t}, t \geq 0 \).

**Proof** Consider the integral equation:

\[
S(t) = T(t) + \int_0^t T(t-s) B S(s) \, ds
\]

We are going to show:
1° For any \( \delta > 0 \), \( M \) \( \| B \| \), eq (1)

Let a unique solution in the Banach space

\[
C(\delta) = \left\{ u \in \mathcal{C}(\mathbb{R}^+, B(x, x)) \mid \sup_{t \geq 0} e^{-\delta t} \| U(t) \|_{B(x, x)} < \infty \right\}
\]

and \( \| S(t) \| \leq M e^{(\omega + M \| B \|) t} \)

2° The solution in part 1° is a semi-group of operators on \( X \) with infinitesimal generator

\[ A + B : \mathcal{D}(A) \to X \]

Proof of 1° Let \( K : C(\delta) \to C(\delta) \)

\[
K U(t) = \int_0^t T(t-s) L U(s) \, ds.
\]

where \( C(\delta) \) is endowed with the norm:

\[
\| U(t) \|_\delta = \sup_{t \geq 0} e^{-\delta t} \| U(t) \|_{B(x, x)}
\]
There \( K \) is a contraction:

\[
C^{-\delta t} \| (KU_1)(t) - (KU_2)(t) \|_{B(x,x)} \leq C^{-\delta t} \int_0^t m e^{\omega(t-s)} \| U_1 \|_{L^1} e^{\delta s} \| U_1 - U_2 \| \, ds
\]

\[
\leq \| U_1 - U_2 \| \frac{m \| B U \|}{\delta - \omega} (1 - e^{(\omega - \delta)t})
\]

\[
\Rightarrow \| KU_1 - KU_2 \| \leq \frac{m \| B U \|}{\delta - \omega} \| U_1 - U_2 \|
\]

\( KU(t) = T(t) + KU(t) \) is also a contraction and has a unique fixed point:

\[
S = KU = T + KS \Rightarrow \quad S(t) = T(t) + \int_0^t T(t-s) \| B \| S(s) \, ds.
\]

Apply the \( B(x,x) \) norm to the above:

\[
\| S(t) \| \leq \| T(t) \| + \int_0^t \| T(t-s) \| \| B \| \| S(s) \| \, ds
\]

\[
\leq m C e^{\omega t} + \int_0^t m e^{\omega(t-s)} e^{\delta s} \| B \| \| S(s) \| \, ds
\]
By Generalized Gronwall inequality

\[(S(t)u) \leq x(t) \text{ where } x(t) \text{ is the sol of}\]

\[
\begin{cases}
  x(0) = 1
  \\
x'(t) = (w + m \|u\|) x(t)
\end{cases}
\]

\[\Rightarrow x(t) = \exp(\int_0^t (w + m \|u\|) s)\]

\[\text{Proof of 2° Semigroup properties:}\]

(i) \(S(0) = T(0) = I_d\)

(ii) \(S(t)S(t_0) = S(t+t_0) \quad \forall \ t, t_0 \geq 0\)

Fix \(t_0 > 0\), multiply (i) to the right by \(S(t_0)\).
Then \(V(t) = S(t)S(t_0) \quad t \geq 0\) satisfies:

(2) \(V(t) = T(t)S(t_0) + (K V(t_0)\)

But from (1) with \(t \mapsto t+t_0\) and \(V(t) = S(t+t_0)\), \(t \geq 0\)
we have
\[ (3) \quad \forall \mathbf{v}(t) = T(t+\theta_0) + \int_0^{t+\theta_0} T(t+\theta_0-s) B \mathbf{s}(s) \, ds \]

\[ = T(t) T(\theta_0) + \int_0^{t} T(t) T(\theta_0-s) B \mathbf{s}(s) \, ds \]

\[ + \int_0^{t+\theta_0} T(t+\theta_0-s) B \mathbf{s}(s) \, ds \]

\[ = T(t) \mathbf{s}(\theta_0) + \int_0^{t} T(t-s') B \mathbf{s}(s'+\theta_0) \, ds' \]

\[ = T(t) \mathbf{s}(\theta_0) + (K \mathbf{v})(t) \]

So \( \mathbf{v} \) and \( \mathbf{w} \) satisfy the same equation involving the contraction \( K \). By uniqueness, (ii) is satisfied.

For any \( x \in X \) we have

\[ \lim_{t \to 0} \frac{1}{t} \int_0^t T(t-s) B \mathbf{s}(s) x \, ds = Bx \]

Indeed for \( t > 0 \) such that \( \| \mathbf{s}(s) x - x \| < \varepsilon \) \( \forall 0 \leq s \leq t \)

\[ \| \frac{1}{t} \int_0^t T(t-s) B(s(s) x - x) \, ds \| \leq \frac{1}{t} \int_0^t \| \mathbf{e}^{\theta_0 s} \| \| B \| \| x \| \, ds \]

\[ \leq M e^{\theta_0 \varepsilon} \]
Hence
\[
\lim_{t \to 0} \frac{1}{t} \int_0^t T(t-s) B \xi(s) \, ds = \lim_{t \to 0} \frac{1}{t} \int_0^t T(t-s) B x \, ds
\]
\[
= B x
\]

Waw
\[
\frac{S(t) x - x}{t} = \frac{T(t) x - x}{t} + \frac{1}{t} \int_0^t T(t-s) B S(s) x \, ds
\]
and \( \lim_{t \to 0} \frac{S(t) x - x}{t} \) exists if \( \lim_{t \to 0} \frac{T(t) x - x}{t} \) exists and
\[
\lim_{t \to 0} \frac{S(t) x - x}{t} = \lim_{t \to 0} \frac{T(t) x - x}{t} + B x
\]
Here \( A + B : D(A) \to X \) is the infinitesimal generator of \( S \).

The theorem is completely proven! QED
**Corollary:** If $A$ is the infinitesimal generator of the $\mathfrak{C}$-semigroup of operators $T(t): X \to X$, $\|T(t)\| \leq M e^{\omega t}$ \( t \geq 0 \), and $B: X \to X$ is linear and bounded then the $\mathfrak{C}$-semigroup generated by $A + B$, $S(t)$ \( t \geq 0 \) satisfies:

\[
\|S(t) - T(t)\| \leq M e^{\omega t} (e^{\mu_{11}B(t)} - 1)
\]

**Proof:** From (1) we have

\[
\|S(t) - T(t)\| \leq \int_0^t \|T(t-s)\| \|B\| \|S(s)\| \, ds
\]

\[
\leq \int_0^t M e^{\omega(t-s)} \|B\| M e^{\mu_{11}B(t-s)} \, ds
\]

\[
= M e^{\omega t} \int_0^t M e^{\mu_{11}B(t-s)} \, ds
\]

\[
= M e^{\omega t} (e^{\mu_{11}B(t)} - 1) \quad \text{Q.E.D}
\]
Theorem (perturbations of infinitesimal generators of contraction semigroups) Let $A$ be the infinitesimal generator of a $C_0$ semigroup of contractions. Let $B$ be dissipative satisfying $D(B) \supset D(A)$ and

$$
\|Bx\| \leq \alpha \|Ax\| + \beta \|x\| \quad \text{for } x \in D(A)
$$

where $0 \leq \alpha < 1$ and $\beta > 0$. Then $A + B$ is the infinitesimal generator of a $C_0$ semigroup of contractions.

Proof. By Lumer–Phillips:

$$
D(A) = X
$$

\forall x \in D(A), \forall \bar{x} \in F(x) \quad Re \langle Ax, \bar{x}^* \rangle \leq 0

and $\text{Rouge}(I - A) = X$.

Consequently:

$$
D(A + B) = D(A) = X
$$

\forall x \in D(A + B) = D(A) \quad \forall \bar{x} \in F(x) : \quad Re \langle Bx, \bar{x}^* \rangle \leq 0

\Rightarrow Re \langle (A + B)x, \bar{x}^* \rangle \leq 0 \quad \Rightarrow A + B \text{ is dissipative}

It remains to show that $\text{Rouge}(I - A - B) = X$ after which Lumer–Phillips implies the conclusion.
Lemma: Let $A, B$ be linear operators such that 
\[ \|B(x)\| \geq \|x\|, \] 
and $A + B$ is dissipative for all $0 \leq t \leq 1$. If 
\[ \|B(x)\| \leq d \|Ax\| + \beta \|x\| \text{ for } x \in D(A) \] 
where $0 < d < 1$, $\beta > 0$, and for some $t_0 \in [0, 1]$ 
\[ \text{Range}(I - (A + t_0 B)) = X \text{ then for all } t \in [0, 1] \] 
\[ \text{Range}(I - (A + t B)) = X \]

Proof of Lemma: It suffices to show that 
\[ \exists \delta > 0 \text{ independent of } t \text{ such that} \] 
\[ \text{Range}(I - (A + t B)) = X \quad \forall t \in [0, 1], \|B - B_0\| \leq \delta \]

The result for $t \in [0, t_0]$ follows by applying the local result for $t_0 \pm \delta$, $t_0 \pm 2\delta$, ...

$A + t_0 B$ dissipative $\Rightarrow \| (I - (A + t_0 B)) x \| \geq \|x\| \quad \Rightarrow \quad$ 
\[ \text{Range}(I - (A + t_0 B)) = X \]

$\left( I - (A + t_0 B) \right)^{-1} = \mathcal{R}(t_0) \text{ exists and } \| \mathcal{R}(t_0) \| \leq 1$

Next we show that $\mathcal{B} R(t_0)$ is bounded:
\[ \|Bx\| \leq d \|Ax\| + \beta \|x\| \leq d \| (A + t_0 B) x \| + \alpha_0 \|Bx\| + \beta \|x\| \] 
\[ \Rightarrow \|Bx\| \leq \frac{d}{1 - d} \| (A + t_0 B) x \| + \frac{\beta}{1 - d} \|x\| \] 
\[ (A + t_0 B) \mathcal{R}(t_0) = \mathcal{R}(t_0) - I \]
\[ \Rightarrow \| \mathcal{B} \mathcal{R}(t_0) x \| \leq \frac{d}{1 - d} \| \mathcal{B} \mathcal{R}(t_0) - I \| \|x\| + \frac{\beta}{1 - d} \| \mathcal{B} \| \|x\| \]
\[ \| \mathbf{B} \mathbf{R}(b_0) \mathbf{x} \| \leq \frac{2d + \beta}{1 - 2} \| \mathbf{x} \| \]

**Wran**

\[
\begin{align*}
I - (A + tB) &= I - (A + b_0 I) + (b_0 - t)B \\
&= (I + (b_0 - t)BR(b_0))(I - (A + b_0 I))
\end{align*}
\]

\[
\text{if } 0 \leq \frac{1}{2} \frac{1 - 2}{2d + \beta}
\]

So for \( |t - b_0| \leq \delta \leq 1 \frac{1 - 2}{2d + \beta} \)

\[
\text{Range}(I - (A + tB)) = X.
\]

This finishes the lemma and the proof of the theorem Q.E.D.

**Remark.** In general one cannot allow \( d = 1 \) in the previous theorem or lemma. However if \( X \) is reflexive (or \( B^* \) has dense domain in \( X^* \)) then under the assumptions of the previous theorem with \( d = 1 \) we have that \( A + tB \) is the infinitesimal generator of a semigroup of contractions.
3.4 The abstract Cauchy problem (the bounded operator case)

Consider $X$ a Banach space and $B : X \to X$ linear and bounded.

(i) The homogeneous Cauchy problem

$$\begin{cases}
\frac{du}{dt} = Bu, & t > 0 \\
v(0) = x \in X
\end{cases} \quad (1)$$

**Def.** $u : [0,T) \to X$ continuous, continuously differentiable on $(0,T)$ and satisfying $(1)$ is called a classical solution of $(1)$.

**Lemma (integral form)** The following are equivalent:

(i) $u$ is a classical solution of $(1)$ on $[0,T)$

(ii) $u : [0,T) \to X$ is continuous and satisfies

$$u(t) = x + \int_0^t Bu(s) \, ds \quad \forall 0 \leq t < T.$$
Proof \((i) \Rightarrow (ii)\) by integration (Riemann integrals on Banach spaces)
\((ii) \Rightarrow (i)\).

\(W : X \to X\) is continuous
\(\Rightarrow \) \(B u : [0, T] \to X\) is continuous
\(\Rightarrow \) \(t \to \int_0^t B u(s) ds\) is differentiable on \([0, T]\)
(by properties of Riemann integrals)

Differentiating \((2)\) we get
\[
\frac{du}{dt} = Bu(t) \quad t > 0
\]

Since right hand side is continuous on \([0, T]\), we get that \(u\) is a classical solution of \((1)\).

Remarks
1. \((ii) \Rightarrow (i)\) is not valid in general for unbounded operators \(B\).
2. Proof of \((ii) \Rightarrow (i)\) shows that classical solutions are actually continuously differentiable on \([0, T]\).
**Theorem:** (1) has a unique classical solution given by the uniform continuous semigroup of bounded operators generated by B:

\[ u(t) = e^{tB} x \quad t \geq 0 \]

**Proof:** Uniqueness: Let \( u_1 : [0, T_1) \rightarrow X \)

\( u_2 : [0, T_2) \rightarrow X \)

be two classical solutions. We show that

\[ u_1 = u_2 \text{ on } [0, T_1) \cap [0, T_2). \]

WLOG we can assume \( T_1 \leq T_2 \)

**Claim.** For any \( T \leq T \) (2) has a unique solution on

\[ C([0, T], X) = \{ \sigma : [0, T] \rightarrow X \mid \sigma \text{ continuous} \} \]

Indeed embed \( C([0, T], X) \) with the Bielecki norm:

\[ \| \sigma \|_p = \sup_{0 \leq t \leq T} e^{-\delta t} \| \sigma(t) \| \]
with $\delta > \|B\|$ fixed. Then 

$$C([0,\infty), X), \|\cdot\|_\infty)$$

is a Banach space (with the norm equivalent with the standard supremum norm).

Let $K : C([0,\infty), X) \to C([0,\infty), X)$

$$(Ku)(t) = \int_0^t L \, v(s) \, ds$$

then $K$ is well defined and is a contraction with Lipschitz constant $\|B\|/\delta$;

$$\|Ku - K\nu\|_\infty \leq \frac{\|B\|}{\delta} \|u(t) - \nu(t)\|_\infty.$$

(2) is equivalent with

$$u = x + Ku$$

which by contraction principle has a unique solution in the space of continuous functions.

The claim is proven.
Now \( U_1, U_2 \) are continuous \( C^\infty \) s of (2) on \([0, T_1]\), hence on \([0, \tilde{T}]\) for all \( \tilde{T} < T_1 \).
So they must coincide with the unique sol of (2) on \([0, \tilde{T}]\). Hence

\[
U_1(t) = U_2(t) \quad \forall \ 0 \leq t \leq \tilde{T}, \ \forall \tilde{T} < T_1
\]

\[\Rightarrow U_1 = U_2 \text{ on } [0, T_1]. \text{ Uniqueness is proven!} \]

Existence: \( U(t) = e^{tb}x \) is a classical solution of (1) on \([0, \infty)\) follows from the properties of \( e^{tb} \), see Lecture 12. QED

(iii) The inhomogeneous problem

Consider

\[
\begin{cases}
\frac{dU}{dt} = bU + f(t) \quad &t > 0 \\
U(0) = x \in X
\end{cases}
\]

where \( b : X \rightarrow X \) is bounded and \( f : (0, T) \rightarrow X \).

The definition of classical solutions is as before
Theorem (i) If \( u \) is a classical solution of (3) and \( f \in L^1([0,T), X] \) then \( u \) satisfies:

\[
(4) \quad u(t) = e^{tB} x + \int_0^t e^{(t-s)B} f(s) \, ds \quad \text{for } 0 \leq t \leq T
\]

(ii) If \( f \in L^1([0,T), X] \cap C([0,T), X] \) then (4) is the classical solution of (3).

Proof (i) Consider \( 0 \leq t \leq T \). Consider

\[
g : [0,t] \to X \quad ; \quad g(s) = e^{(t-s)B} u(s)
\]

Since \( u(s) \) is a classical solution hence differentiable on \((0,t]\) we have for \( 0 < s < t \):

\[
\frac{dg}{ds} = -e^{(t-s)B} u(s) + e^{(t-s)B} u'(s)
\]

\[
= e^{(t-s)B} u(s) + e^{(t-s)B} \left[ B u(s) + f(s) \right]
\]

\[
= e^{(t-s)B} f(s)
\]

\[
\in L^1(0,T)
\]

Integrating on \((0,T)\) we get
\[ y(t) - y(0) = \int_0^t e^{(t-s)B} \varphi(s) \, ds \]

\[ u(t) - e^{tB} x = \int_0^t e^{(t-s)B} \varphi(s) \, ds \]

(ii) By (i) the classical solution is unique.

\[ u(t) \text{ given by } (4) \text{ is evidently continuous on } [0, T] \text{ and } u(0) = x. \]

Since \( f \) is continuous on \((0, T)\),

\[ \Rightarrow t \to \int_0^t e^{(t-s)B} \varphi(s) \, ds \text{ is differentiable on } (0, T) \]

and

\[ \frac{d}{dt} \int_0^t e^{(t-s)B} \varphi(s) \, ds = f(t) + B \int_0^t e^{(t-s)B} \varphi(s) \, ds \]


\[ \text{continuity, } u(0, T) \]

Hence \( u(t) \) given by (4) is continuously differentiable on \((0, T)\) and

\[ \frac{du}{dt} = B e^{tB} x + f(t) + B \int_0^t e^{(t-s)B} \varphi(s) \, ds \]

\[ = B \left( e^{tB} x + \int_0^t e^{(t-s)B} \varphi(s) \, ds \right) + f(t) \]

\[ = B u(t) + f(t) \]

\[ \therefore \text{QED} \]
Remark. If \( f(t) \) is not continuous on \((0, T)\),
\( \text{then there are no classical solutions.} \)

Def. If \( f \in L^1([0, T], X) \) then \( U \) given by
\( \text{(4)} \) is called a weak solution.

Theorem. (i) If \( U \) is a weak solution of
\( \text{(3)} \) then \( \exists \{f_n, y_n \in L^1([0, T], X) \} \) \( f_n \to f \)
and \( y_n \to y \) in \( X \) such that
\[
\begin{align*}
\frac{dU_n}{dt} &= BU_n + f_n(t) \\
U_n(0) &= y_n
\end{align*}
\]
\( \text{has classical solutions and } U_n \to U \text{ in } C([0, T], X). \)

(ii) If \( U \) is a weak solution of \( \text{(3)} \)
then \( \frac{dU}{dt} \) exists almost everywhere \( \frac{dU}{dt} \in L^1([0, T], X) \)
and \( \frac{dU}{dt} = BU + f(t) \) a.e. (in other
\( \text{words, } U \text{ is a strong solution of } \text{(3)}. \)
Proof (i) \( f \in C^1([0,\tau], X) \)

\[ \Rightarrow \exists f_n \in C([0,\tau], X) \text{ such that } \]

\[ f_n \overset{L}{\to} f \quad (i.e.: \quad \lim_{n \to \infty} \int_0^\tau \| f(t) - f_n(t) \| \, dt = 0). \]

Then by previous theorem part (ii):

\[ \begin{cases} \frac{d u_n}{dt} = \mathcal{B} u_n + f_n(t) \\ u_n(0) = x \end{cases} \]

The unique solution \( u_n \) given by:

\[ u_n(t) = e^{t\mathcal{B}} x + \int_0^t e^{(t-s)\mathcal{B}} f_n(s) \, ds \]

Then

\[ |u_n(t) - u(t)| \leq \int_0^t e^{(t-s)\|\mathcal{B}\|} \| f_n(s) - f(s) \| \, ds. \]

\[ \leq e^{\|\mathcal{B}\| \tau} \| f_n - f \|_{L^1([0,\tau], X)} \quad Q.E.D. \]
(ii) \( e^{tB} \int_0^t e^{-sB} f(s) \, ds \) is differentiable a.e.

with derivative:

\[ B e^{tB} \int_0^t e^{-sB} f(s) \, ds + e^{tB} e^{-tB} f(t) \text{ a.e.} \]

Since \( e^{tB} \) is differentiable everywhere \( \Rightarrow \) given

\[ \log C_1 \text{ is differentiable a.e.:} \]

\[ \frac{dv}{dt} = B e^{tB} x + B e^{tB} \int_0^t e^{-sB} f(s) \, ds + f(t) \]

\[ = B v(t) + f(t) \text{ a.e.} \quad \square \]