3. Semigroups of Operators and Nonlinearities


3.3. Perturbations of infinitesimal generator.

3.4. The Abstract Cauchy Problem.

3.4.1. An application to the linear Schrödinger equation.

3.5. An application to the nonlinear Schrödinger equation.


Def (Semigroup of operators): \( X \) Banach space.
\( T(t): X \to X \) linear and bounded \( 0 \leq t < \infty \) is called a semigroup if:

(i) \( T(0) = I_d \)

(ii) \( T(t+s) = T(t)T(s) \quad \forall t, s > 0 \)
Def: A semigroup of bounded linear operators $T(t)$ is called:

(a) uniformly continuous if

$$\lim_{t \to 0} ||T(t) - I_d|| = 0$$

(b) (strongly) continuous if

$$\lim_{t \to 0} ||T(t)x - x|| = 0 \quad \forall x \in X$$

Remark: By (ii) we have:

(a) For uniformly continuous semigroup $T(t)$

$$\lim_{t \to s, \quad t > s} ||T(t) - T(s)|| = 0$$

(b) For continuous semigroups

$$\lim_{t \to s, \quad t > s} ||T(t)x - T(s)x|| = 0$$
Def: The linear operator \( A : D(A) \subseteq X \rightarrow X \) defined by:

\[
D(A) = \{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \}
\]

\[
Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt} \bigg|_{t=0}
\]

is called the infinitesimal generator of the semigroup \( T(t) \).

Theorem 1

1° If \( T(t) \) is a uniformly continuous semigroup then its infinitesimal generator is bounded.

2° Any linear bounded operator \( A : X \rightarrow X \) generates a uniformly continuous semigroup of operators.

Proof 1° Since \( \lim_{s \to 0} s^{-1} \int_0^s T(s) ds. = \text{Id} \).

We can choose \( s \) small enough so that

\[
\| \text{Id} - s^{-1} \int_0^s T(s) ds \|_g(x,x) < 1
\]
\[
\begin{align*}
\Rightarrow \quad & \quad \text{Id} - (\text{Id} - g^{-1} \int_0^S T(s) \, ds) \text{ is invertible} \\
\Rightarrow \quad & \quad g^{-1} \int_0^S T(s) \, ds \text{ is invertible for } S > 0 \text{ small.} \\
\end{align*}
\]

Fix such a $S \Rightarrow \int_0^S T(s) \, ds$ is invertible.

\[
\begin{align*}
(T(h) - \text{Id}) \int_0^S T(s) \, ds &= \int_0^{S+h} T(s) \, ds - \int_0^S T(s) \, ds \\
&= \int_0^h T(s) \, ds - \int_0^S T(s) \, ds \\
&= \int_S^{S+h} T(s) \, ds - \int_0^h T(s) \, ds. \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \quad & \quad h^{-1} (T(h) - \text{Id}) = (h^{-1} \int_S^{S+h} T(s) \, ds - h^{-1} \int_0^h T(s) \, ds) \\
& \quad \quad \times \left(\int_0^h T(s) \, ds\right)^{-1} \\
\Rightarrow \quad & \quad \lim_{h \to 0} \overline{h^{-1} (T(h) - \text{Id})} = \overline{(T(S) - \text{Id}) \left(\int_0^S T(s) \, ds\right)^{-1}} \\
& \quad \quad \text{bounded linear operator.} \\
\end{align*}
\]

\[
\begin{align*}
2^o \quad & \quad \text{Define } T(h) = e^{thA} = \sum_{n=0}^{\infty} \frac{(thA)^n}{n!} \\
& \quad \quad - I + thA + \frac{th^2A^2}{2} \\
\text{The RHS is absolutely convergent in } \mathcal{B}(X, X). 
\end{align*}
\]
Then, $T(t)$ is a semigroup of operators (check) and

$$T(t) - \text{Id} = \sum_{n=1}^{\infty} \frac{t^n A}{n!} = tA \left( \sum_{n=0}^{\infty} \frac{t^n A}{(n+1)!} \right)$$

$$= \int T(t) - \text{Id} \leq |t| |A| \exp \left( \|A(x,x)\|_{B(x,x)} t \right)$$

$$= \int T(t) - \text{Id} \leq \frac{t |A|}{B(x,x)} \rightarrow 0$$

Moreover,

$$\| T(t) - \text{Id} \|_{B(x,x)} \leq \left| \frac{t A^2 + t A^3 + \ldots}{2 + 3!} \right| \leq |A| e^{\|A\|_B t}$$

$$\Rightarrow \lim_{t \to 0} \frac{T(t) - \text{Id}}{t} = A \Rightarrow A$$ is the infinitesimal generator of $T(t) = e^{tA}$. QED

Corollary 1

If $T(t)$ is a uniformly continuous semigroup of operators then

1) \exists \omega > 0 such that $\|T(t)\| \leq e^{\omega t}$

2) \exists A \in B(x,x) such that $T(t) = e^{tA}$

3) $t \to T(t)$ is differentiable in norm and

$$\frac{d}{dt} T(t) = AT(t) = T(t)A$$
Theorem 2. If \( T(t) \) is a continuous semigroup of operators then \( \exists \, \delta > 0 \) and \( M > 1 \) such that

\[
\| T(t) \| \leq Me^{-\delta t} \quad \forall \, t \geq 0
\]

Proof. \( \exists \, \delta > 0 \) and \( M > 1 \) such that

\[
\| T(t) \| \leq M \quad \forall \, 0 \leq t \leq \delta
\]

Otherwise \( \exists \, \eta \in Y \) such that

\[
\| T(\eta) \| > M
\]

But \( \| T(\eta) \| \) is bounded for every \( \eta \in X \) because

\[
\| T(\eta) \| X \to X \quad \text{is bounded} \]

By Hahn-Banach principle \( \exists \, \omega \in X \) such that \( \| T(\omega) \| \) is bounded contradicting:

\[
\| T(\omega) \| \geq M
\]

Choose \( \omega = \frac{\eta}{M} \geq 0 \).

For \( t \geq 0 \) \( \exists \, \mu \in \mathbb{R} \), \( \varepsilon > 0 \) : \( t = M\delta + \varepsilon \), \( \varepsilon < \delta \)

\[
\Rightarrow \| T(t) \| = \| T(\varepsilon) T(\mu) \| = \| T(\varepsilon) \| \| T(\mu) \| \leq M^{\varepsilon} M^{\mu} \leq M^\mu \| T(\varepsilon) \| = M^e e^{\varepsilon t}
\]

Theorem 3. If \( T(t) \) is a continuous semigroup of operators with infinitesimal generator \( A \) then

a) \( \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} T(s)x \, ds = T(t)x \quad \forall \, x \in X \)
b) \( \int_0^t T(s) x \, ds \in D(A) \quad \forall x \in X, \ t > 0 \) and
\[
A \left[ \int_0^t T(s) x \, ds \right] = T(t)x - x.
\]

c) \( \forall x \in D(A) \Rightarrow T(t)x \in D(A) \) and
\[
\frac{d}{dt} T(t)x = A T(t)x = T(t)A x \quad \frac{d}{dt}
\]

\( \forall x \in D(A) \)

\[
T(t)x - T(s)x = \int_s^t T(s)A x \, ds = \int_s^t A T(s)x \, ds.
\]

**Proof:** a) Follows from continuity of \( t \mapsto T(t)x \)

b) For \( x \in X, \ h > 0 \) we have

\[
\frac{T(h)-I}{h} \int_0^t T(s)x \, ds = \frac{1}{h} \int_0^{t+h} T(s)x \, ds - \frac{1}{h} \int_0^t T(s)x \, ds
\]

\( h \to 0 \) proves b).

c) For \( x \in D(A), \ h > 0 \) we have
\[
\frac{\Gamma(u) - \Gamma(0)}{u} = \frac{1}{u} \lim_{u \to 0} \Gamma(u) = T(0) x \rightarrow T(t) A x
\]

\Rightarrow T(t) x \in \mathcal{D}(A) \text{ and}

\[
\lim_{u \to 0} \frac{T(t + u) x - T(t) x}{u} = A T(t) x = T(t) A x
\]

But for \( t > 0 \):

\[
\lim_{u \to 0} \left[ \frac{T(t) x - T(t-h) x}{u} - T(t) A x \right] =
\]

\[
= \lim_{u \to 0} T(t-h) \left[ \frac{T(h) x - x - A x}{u} \right] +
\]

\[
\lim_{u \to 0} \left[ T(t-h) A x - T(t) A x \right] = 0
\]

So for \( t > 0 \):

\[
\lim_{u \to 0} \frac{T(t + u) x - T(t) x}{u} = T(t) A x = A T(t) x
\]

(3) is finished and by integrating it we get

(4) QED
Corollary If $A$ is the infinitesimal generator of the continuous semigroup of operators $T(t)$ then

$$D(A) = X$$  and $A$ is closed.

Proof For $x \in X$ set $x_n = \frac{1}{n} \int_0^t T(s)x \, ds$

by (b) $x_n \in D(A)$, by (a) $x_n \to x \Rightarrow \overline{D(A)} = X$

Let $x_n \in D(A)$ $x_n \to x \in X$ and $A x_n \to y \in X$. Then

$$T(t)x_n - x_n = \int_0^t T(s)A x_n \, ds.$$  

Converges to $T(t)x - x$ converges  

$$= \lim_{t \to 0} T(t)x - x = \int_0^t T(s)y \, ds.$$  

$$= \lim_{t \to 0} T(t)x_n - x \text{ exists and } = y$$  

$$= x \in D(A) \text{ and } A x = y \Rightarrow A \text{ closed.}$$  

QED
Remark: If $T(t)$ is a continuous semigroup with infinitesimal generator $A$ and $\{DCA^n\}_{n=1}^{\infty}$ is the closure of $A^n$, then
$$\bigcap_{n=1}^{\infty} DCA^n = \varnothing.$$

Proof: $\gamma = \{ \int_0^\infty \varphi(s) T(s)x \, ds \mid x \in X, \varphi \in C^0(\mathbb{R}^+), \gamma \}$
Then $\gamma \subseteq \bigcap_{n=1}^{\infty} D(n)$ and $\gamma = \varnothing$.

Theorem 4: If $T(t)$ and $S(t)$ are continuous semigroups of operators with the same generator then $T(t) \equiv S(t)$.

Proof: Let $A$ be their common infinite infinitesimal generator and $x \in D(A)$. Then for $t > 0$, $0 < s < t$ and $s \to T(t-s)S(s)x$ is differentiable and
$$\frac{d}{ds} T(t-s)S(s)x = -T(t-s)AS(s)x + T(t-s)A S(s)x = 0.$$

$\Rightarrow$ $s \to T(t-s)S(s)x$ is constant $= C$ for $s = 0$ and $s = t: T(t)x = S(t)x \equiv C.$