Summary

1. Organizational issues

2. Overview of the material to be covered

1. Organizational issues

- First day handout

2. Overview of the material to be covered

1. Bifurcation Theory in PDE's

An example from ODE's:

\[
\frac{dx}{dt} = F(x, x) \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}
\]

\[F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\] is \(C^1\) and:

\[F(x, 0) = 0 \quad \forall x \in \mathbb{R}\]
We are looking for equilibrium solutions:

\[ F(x, n) = 0 \]

0 is an equilibrium for all \( n \in \mathbb{R} \).

**Case I** \( D_x F(x_0, 0) \) (which is a \( n \times n \) matrix) is invertible.

Then, by implicit function theorem, (1) has a unique solution \( x = 0 \) for \( |n - n_0| \) and \( |x| \) sufficiently small.

**Case II** \( D_x F(x_0, 0) \) is not invertible.

Then (1) may have other solutions besides \( x = 0 \) (which "bifurcate" from \( x = 0 \) at \( n = n_0 \)).

Assume \( \ker D_x F(x_0, 0) = \text{span} \{ v, y \} \).

Then any \( x \in \mathbb{R}^n \) can be written in a unique way:

\[ x = \alpha v + x_1, \quad x_1 \in \text{Range } D_x F(x_0, 0) \]
Similarly:

\[ F(x, \xi) = F_1(x, \xi) \psi + F_2(x, \xi) \]

with \( F_1: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) and \( F_2: \mathbb{R} \times \mathbb{R}^n \to \text{range } D_\xi F(\lambda_0, 0) \)

(1) becomes:

\[ G(x, \alpha, \xi) = F_2(x, \alpha \psi + x_1) = 0 \]

\[ F(x, \alpha \psi + x_1) = 0 \]

Since \( G(x_0, 0, 0) = 0 \) and \( D_\xi G(x_0, 0, 0) = D_\xi F(\lambda_0, 0) \) is

invertible, the first eq. by implicit function theorem, gives a \( C^1 \) surface of solutions:

\[ x_1 = x_1(x, \alpha) \text{ with } x_1(x_0, 0) = 0 \]

Replacing in the second eq.

\[ G_1(x, \alpha) = F_1(x, \alpha \psi + x_1(x, \alpha)) = 0 \]

with \( G_1(x_0, 0) = 0 \)

\[
\frac{\partial G_1}{\partial \lambda}(x_0, 0) = \frac{\partial F_1}{\partial \lambda}(x_0, 0) + \nabla_\xi F_1(x_0, 0) \cdot \frac{\partial x_1}{\partial \lambda} = \frac{\partial F_1}{\partial \lambda}(x_0, 0) = 0
\]
because \( F(x, 0) = 0 \Rightarrow F_1(x, 0) = 0 \Rightarrow \frac{\partial F}{\partial x}(x_0, 0) = 0 \)

also:

\[
\frac{\partial G_i}{\partial a}(x_0, 0) = \nabla F_i(x_0, 0) \cdot (u + \frac{\partial x}{\partial a}) = 0
\]

Hence \( G_1 : \mathbb{R}^2 \rightarrow \mathbb{R} \) is zero at \((x_0, 0)\) and has zero gradient. We will see (Morse lemma) that if \( G_1 \in C^3 \) and its Hermitian at \((x_0, 0)\)

\[
\nabla G_i(x_0, 0) = \begin{bmatrix}
\frac{\partial^2 G_i}{\partial n^2}, & \frac{\partial^2 G_i}{\partial n \partial a} \\
\frac{\partial^2 G_i}{\partial n \partial a}, & \frac{\partial^2 G_i}{\partial a^2}
\end{bmatrix}
\]

(where all derivatives are calculated at \((x_0, 0)\))

in nondegenerate i.e.,

\[\det \nabla G_i(x_0, 0) \neq 0\]

then there is a near identity change of variables \( f : \mathbb{R} - x_0 \) a small:

\[
(x, a) \rightarrow (\tilde{x}, \tilde{a})
\]
such that

\[ G(\tilde{\alpha}, \tilde{\lambda}) = \begin{bmatrix} \tilde{\lambda} - \lambda_0 \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\
\tilde{\lambda} - \lambda_0 \end{bmatrix} \]

hence \( \Theta = G(\tilde{\lambda}, \tilde{\alpha}) \) is an quadratic equation that can be shown to have 2 solutions:

\[ \tilde{\alpha} = 0, \tilde{\lambda} \text{ arbitrary} \Rightarrow \alpha = 0, \lambda \text{ arbitrary} \]

and

\[ \tilde{\lambda} = \lambda(\tilde{\alpha}) = \lambda = \lambda(\alpha). \] This is the new solution curve: \( \tilde{x} = \alpha \tilde{u} + x_i(\alpha, \lambda(\alpha)) \).

Questions: How we generalize such technique to the case when \( F \) contains differential operators, for example:

\[ F(\lambda, x) \rightarrow \tilde{F}(\lambda, \tilde{u}) = [-A + V(x)] \tilde{u} + |\tilde{u}|^2 \tilde{u} + \lambda \tilde{u} \]

\[ F: \mathbb{R} \times H^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \]

We will need first to generalize the implicit function theorem to abstract Banach spaces.
2. Applications of semigroup of operators in PDE's.

Start from a system of ODE's:

\[
\begin{align*}
\frac{dx}{dt} &= Ax + F(x), \quad x \in \mathbb{R}^n \\
\phi(0) &= x_0
\end{align*}
\]

(2)

$A$ is an $n \times n$ matrix and $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Duhamel's principle says that any C' solution of:

\[
\phi(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} F(\phi(s)) \, ds
\]

is also a solution of (2). This can be checked by differentiation.

(3) is used to show asymptotic stability of the zero solution under the hypotheses:

(i) $\lambda$-values of $A$ have real part negative,

(ii) $F(0) = 0$, $DF(0) = 0$. 

In the PDE case $A$ is replaced by a partial differential operator, for example:

$$\frac{du}{dt} = \Delta u + F(u) \quad U: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\Delta u = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

To use (3) we need to make sense of

$$e^{At}$$

which is the subject of theory of semigroup of operators.

3. Variational methods in PDE's:

An ODE example: $x(t) \in \mathbb{R}^n$

$$\frac{d^2 x}{dt^2} = F(x) = -\nabla_x V, \quad V: \mathbb{R}^n \rightarrow \mathbb{R}$$

Define the energy $E: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$E(x,y) = \frac{1}{2} y \cdot y + V(x)$$

Note that $E(x(t), \frac{d^2}{dt^2}(t)) = \text{constant}$
In solutions of the ODE because
\[
\frac{d^2}{dt^2} (x(t), \frac{dx}{dt}) = \frac{dx}{dt}, \frac{d^2x}{dt^2} + \nabla_x V \cdot \frac{dx}{dt} = 0
\]

Moreover any equilibrium of the ODE:
\[
x(t) = x_0 = \text{constant} \implies 0 = \frac{dx}{dt} = y
\]

is a critical point of \( E \):
\[
0 = \nabla_{x,y} E = (\nabla_x V(x_0), y)
\]

and vice versa. Now any isolated local minimum \( E \) gives a stable equilibrium.
Indeed, for any \( \varepsilon > 0 \) let
\[
\overline{E} = \min \left( E, \varepsilon \right)
\]

where \( \varepsilon > 0 \) in such that
\[
E(x,y) > \overline{E}(x_0,0)
\]

for all \( |x-x_0| + |y| < \varepsilon \) and \( y > 0 \) since \( (x_0,0) \) is an isolated minimum of \( E \).

Now let
\[ u = \inf \left\{ \varepsilon(x,y) \mid (x-x_0) + |y| = \varepsilon \right\} \]

Write that the strict inequality is a consequence of compactness of the set

\[ B_{\frac{\varepsilon}{2}} = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x-x_0| + |y| < \frac{\varepsilon}{2} \right\} \]

Continuity of \( \varepsilon \) and \( x \).

Choose \( \delta > 0 \) such that

\[ 0 \leq \varepsilon(x,y) - \varepsilon(x_0,0) < u \quad \forall |x-x_0| + |y| < \delta \]

which exists by continuity of \( \varepsilon \). It follows that initial data \( x(t_0), \frac{dx}{dt}(t_0) \) satisfying

\[ |x(t_0) - x_0| + \left| \frac{dx}{dt}(t_0) \right| < \delta \]

have trajectories \( |x(t) - x_0| \leq \varepsilon \leq \delta \) otherwise \( \varepsilon(x(t), \frac{dx}{dt}) \) must needs be bounded or unbounded since \( u \) is constant. Hence \( x_0 \) is a stable equilibrium.

We will look for generalizations to \( PDE \)'s with Hamiltonian structure.