Math 554
Homework 6
Due December 4, 2008, before class

In what follows $U$ will be a bounded, open set in $\mathbb{R}^n$:

**Problem I** Assume $U$ is also connected. A function $u \in H^1(U)$ is a weak solution of the *Neumann's problem*:

\[-\Delta u = f \quad \text{on } U\]
\[\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial U\]

if, for all $v \in H^1(U)$ :

\[\int_U \nabla u \cdot \nabla v \, dx = \int_U fv \, dx.\]

(i) Explain why this is a good definition of a weak solution;

(ii) Show that for $f \in L^2(U)$, the above problem has a weak solution if and only if

\[\int_U f \, dx = 0.\]

**Problem II** Consider the wave equation:

\[\frac{\partial^2 u}{\partial t^2} = \Delta u + f(t, x) \quad \text{on } t > 0, \ x \in U\]
\[u(t, x) = 0 \quad \text{on } \partial U \text{ for } t > 0\]
\[u(0, x) = u^0(x)\]
\[\frac{\partial u}{\partial t}(0, x) = u^1(x).\]

(i) Define an appropriate strong solution for this equation;

(ii) Find (minimal) hypothesis on $f$, $u^0$, $u^1$ such that the problem has a unique strong solution that can be obtained as a series involving the e-values of the -Laplacian.
Problem III Consider the following Dirichlet problem in non-divergence form:

\[-\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^{2}u}{\partial x_{i}\partial x_{j}} + \sum_{i=1}^{n} b^{i}(x) \frac{\partial u}{\partial x_{i}} + c(x)u = f \quad \text{on } U\]

\[u = 0 \quad \text{on } \partial U.\]

(i) Find sufficient conditions such that the bilinear form induced by the problem is symmetric: \(B[u, v] = B[v, u]\) for all \(u, v \in H^{1}_{0}(U)\). Note that the conditions given in Lecture 16 Remark 1 are not sufficient.

(ii) Show that \(B\) symmetric is equivalent to the symmetry of the operator \(A\) in the proof of Lax-Milgram Theorem;

(iii) Under the hypotheses of Lax-Milgram Theorem, with \(A\) symmetric, show that \((Au, u)\) is a scalar product and \(H\) is a Hilbert space with respect to the norm induced by it.

Problem IV Let \(U = (0, 1) \times (0, 1)\). Consider the Helmholtz problem:

\[-\Delta u + \frac{\partial u}{\partial x_{1}} = \lambda u + f \quad \text{on } U\]

\[u = 0 \quad \text{on } \partial U.\]

(i) For what values of \(\lambda\) the problem has a unique weak solution?

(ii) If the above problem is not uniquely solvable write down explicitly the conditions on \(f\) for which it is solvable. Describe the set of functions by which two weak solutions can differ.