Math 554
Homework 5
Due November 14, 2008, before class

Unless otherwise stated, in what follows $X$ and $Y$ will be normed spaces over the same $K = \mathbb{R}$, or $\mathbb{C}$.

**Problem I** If $L : X \mapsto Y$ is linear and continuous then the adjoint operator is continuous and $\|L^*\| = \|L\|$. 

**Problem II** If $A : X \mapsto X$ is linear and compact then $y^* \in \text{Range}(A^* - \text{Id})$ implies $\ker(A - \text{Id}) \subseteq \ker y^*$. (This is the necessity part of Lemma 2).

**Problem III** Assume $X$ is a reflexive Banach space and $A : X \mapsto X$ is linear and compact. Show directly, using Fredholm alternative, that $y^* \in \text{Range}(A^* - \text{Id})$ is equivalent to $\ker(A - \text{Id}) \subseteq \ker y^*$.

**Problem IV** Assume $A : X \mapsto X$ is linear and compact. Show that the equations:

\[
Ax - x = y, \quad y \in X \quad \quad (1)
\]
\[
A^*x^* - x^* = y^*, \quad y^* \in X^* \quad \quad (2)
\]

are either both uniquely solvable or, if they are solvable they have the same number of (finitely many) independent solutions. In the latter case what condition on $y$ respectively $y^*$ are equivalent with the solvability of (1) respectively of (2).