Problem I
1. State and prove a trace operator theorem of the type
\[ T : W^{m,p}(U) \hookrightarrow W^{1,p}(\partial U). \]
Try to minimize \( m \) and the assumptions on the open set \( U \subset \mathbb{R}^n \).
2. Consider the trace operator:
\[ T : W^{1,2}(U) \hookrightarrow L^2(\partial U), \]
defined in lecture 7 for \( U \subset \mathbb{R}^n \) open, with a bounded, piecewise \( C^1 \) boundary satisfying the segment property and such that any intersection of two \( C^1 \) pieces is transversal. Show that if \( g : \partial U \hookrightarrow \mathbb{R} \) is continuous and piecewise \( C^1 \) on the boundary then there exists a \( f \in W^{1,2}(U) \) such that \( Tf = g \). (suggested) Show that it suffices to have \( g \in C^{0,\alpha}(\partial U), \alpha > \frac{1 + \sqrt{5}}{4} \), for the above conclusion to still hold. This means that \( g \) is continuous on the boundary and there exists a constant \( K > \) such that for any \( x, y \) on the same \( C^1 \) piece of the boundary we have: \( |g(x) - g(y)| \leq K|x - y|^{\alpha} \).

Problem II
Prove that an extension operator \( E : W^{m,p}(U) \hookrightarrow W^{m,p}(\mathbb{R}^n) \) exists for any \( m \in \mathbb{N} \) and \( 1 \leq p < \infty \) when:
1. \( n = 2 \) and \( U = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1\} \);
2. \( n = 3 \) and \( U = \{x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0\} \).

Problem III
Hölder spaces. Here \( U \) is an open set in \( \mathbb{R}^n \).
\[ \|g\|_{C^{0,\gamma}(\overline{U})} = \sup_{x \in \overline{U}} |g(x)| + \sup_{x,y \in \overline{U}, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^{\gamma}} \]
\[ C^{0,\gamma}(\overline{U}) = \{g \in C(U) \mid \|g\|_{C^{0,\gamma}(\overline{U})} < \infty\}. \]
Show that \((C^{0,\gamma}(\overline{U}), \|g\|_{C^{0,\gamma}(\overline{U})})\) is a Banach space. (Suggested) Generalize to \( C^{j,\gamma}(\overline{U}), j = 0, 1, \ldots \).