Summary

5. Applications involving non-symmetric elliptic operators:

5.1. Poisson Eq. (elliptic)
5.2. Heat Eq. (parabolic)
5.3. Wave Eq. (hyperbolic)

5.1. Poisson Eq. and Lax–Milgram Th.

\[ \begin{align*}
-\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} &+ \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f \\
| u |_{\partial U} & = 0
\end{align*} \]

If \( u \in C^2(U) \cap C(\overline{U}) \) is a soln of the above
then, for \( u \in C_c^\infty(U) \) we get:

\[ \begin{align*}
\int_U \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_U \sum_{i=1}^{n} b_i(x) v \frac{\partial u}{\partial x_i} + c(x) u v \, dx = \int_U f u \, dx
\end{align*} \]
Assume $\alpha^{ij} \in \mathcal{W}^{1,\infty}(\Omega)$, $b, c \in L^\infty(\Omega)$ and, for $u, v \in H_0^1(\Omega)$ denote:

\begin{equation}
B[u, v] = \int_\Omega \sum_{i,j} \alpha^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \, dx + \int_\Omega \sum_i b^i(x) \frac{\partial u}{\partial x^i} v \, dx + \int_\Omega c(x) uv \, dx.
\end{equation}

Note that if Eq. in (6) is in divergence form

\begin{equation}
\sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left( \alpha^{ij} \frac{\partial u}{\partial x^j} \right) + \sum_i b^i(x) \frac{\partial u}{\partial x^i} + cu = f
\end{equation}

then $B[u, v] = \int_\Omega \sum_{i,j} \alpha^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \, dx + \int_\Omega \sum_i b^i(x) \frac{\partial u}{\partial x^i} v \, dx + \int_\Omega c(x) uv \, dx$

is well defined for $\alpha^{ij}, b^i, c \in L^\infty(\Omega)$.

Def (weak solution). If $b^i, c \in L^\infty(\Omega)$, $f \in H^{-1}(\Omega)$, and $\alpha^{ij} \in \mathcal{W}^{1,\infty}(\Omega)$ in general or $\alpha^{ij} \in L^\infty(\Omega)$ for Eq. in divergence form, then $u \in H_0^1(\Omega)$ is a weak
solution of (2) if

$$\|B [u, v]\| = \langle f, u \rangle \quad \forall u \in H.$$ (0)

**Theorem (Lax-Phillips):** Let $H$ be a Hilbert space over $K = \mathbb{R}$ or $\mathbb{C}$, let

$$B : H \times H \to K$$

be a bilinear map such that:

(i) **Boundedness:** \( \exists \lambda > 0 \)

$$\|B[u, v]\| \leq \lambda \|u\| \|v\| \quad \forall u, v \in H \quad (4)$$

(ii) **Coercivity:** \( \exists \beta > 0 \)

$$\beta \|u\|^2 \leq \|B[u, v]\| \quad \forall u \in H \quad (5)$$

and $f \in H^*$. Then the equation

$$\|B[u, v]\| = \langle f, u \rangle \quad \forall u \in H$$

has a unique $u \in H$ that depends continuously on $f$. 
Proof: It is nicely done in Evans except the continuous dependence of \( \nu \) on \( f \). This will be only a sketch:

Fix \( v \in H \), then \( \mathcal{B}(\nu, v) : H \to K \) is a linear, bounded functional (boundedness comes from (i)). By Riesz theorem \( \exists! \\lambda \in A^{*} \) such that:

\[
\mathcal{B}(\nu, v) = (\lambda, v) \quad \forall v \in H,
\]

where \((\cdot, \cdot)\) is the scalar product in \( H \).

\( \mathcal{B} \) linear in \( v \), \( \lambda \) \( \mathcal{B} \) is linear and bounded.

\((i) \Rightarrow A \) bijective:

\[
\begin{align*}
\lambda \in H^* & \mapsto \exists! \omega \in H \quad \langle f, v \rangle = (\omega, v) \quad \forall v \in H, \\
\text{and} & \\
\|\omega - \lambda\| & = \|f\|.
\end{align*}
\]

Hence \( \mathcal{B}(\nu, v) = \langle f, v \rangle \quad \forall v \in H \)

\((\ast) \Rightarrow A \nu = \omega \)

Since \( A : H \to H \) is bijective \( \Rightarrow \exists! \nu \in H \) such that \( A \nu = \omega \)

By the open mapping theorem \( \exists C > 0 \) such that
\[ \| u \|_1 \leq C \| u \|_1 = C \| f \|_1 \]

Here \( u \) depends continuously on \( f \).

Remark 1: If \( A \) were symmetric, i.e.,
\[
\alpha \beta = \alpha \beta \quad \text{and} \quad \sum_i \frac{\partial}{\partial x_i} (b_i + \frac{\partial \alpha}{\partial x_i}) = 0
\]
on \( \alpha \beta = \alpha \beta \) and \( \sum_i \frac{\partial}{\partial x_i} b_i = 0 \) then

\[ u, v \rightarrow (A u, v) \] is a new scalar product

on \( W \) which makes \( W \) a Hilbert space and there exists \( \exists ! u + \) its continuous dependence on \( f \) can be inferred from Green Representation The.

Lemma (energy estimate): For \( \beta : H_0^1 \times H_0^1 \rightarrow K \)
defined by \((\beta_\nu)\) or \((\beta_\psi)\) there exists \( L > 0 \)
such that

\[ |B(u, \nu)| \leq L \| u \|_{H_0^1} \| \nu \|_{H_0^1} \quad \text{for } u, \nu \in H_0^1 \]

Proof

\[
\int \left| \alpha \beta \frac{\partial u}{\partial x} \frac{\partial \nu}{\partial x} \right| \, dx \leq \| \alpha \beta \|_{L^2} \left( \frac{\partial u}{\partial x} \right)_{L^2} \left( \frac{\partial \nu}{\partial x} \right)_{L^2}
\]

\[
\int |\alpha \beta \partial x_i u \partial x_i \nu| \, dx \leq \| \alpha \beta \| \left( \frac{\partial u}{\partial x_i} \right)_{L^2} \left( \frac{\partial \nu}{\partial x_i} \right)_{L^2}
\]

\[
\int |\alpha \beta \partial x_i \nu| \, dx \leq \| \alpha \beta \| \left( \frac{\partial \nu}{\partial x_i} \right)_{L^2}
\]

\[
\int |\alpha \beta \partial x_i \partial x_i u \nu| \, dx \leq \| \alpha \beta \| \left( \frac{\partial u}{\partial x_i} \right)_{L^2} \left( \frac{\partial \nu}{\partial x_i} \right)_{L^2}
\]
\[ \int c(x)|u|^2 \, dx \leq \| c \|_{L^\infty} \| u \|_{L^2}^2 \]

Add up \[ \Rightarrow Q \in C \]

Lemma (elliptic estimate) Assume:

\[ (ii) \text{ Uniform ellipticity condition: } \exists \theta > 0 \]
\[ \sum a^{ij}(x) \xi_i \xi_j \geq \theta \left( \sum b_i \xi_i \right)^2 \forall (\xi_i, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \]

then there exists \( \gamma > 0 \) such that:

\[ \exists \beta > 0: \| u \|_{H_0^1}^2 \leq B \| u \|_{L^2} + \gamma \| u \|_{L^2}^2 \]

Proof:
\[ \sum a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \geq \theta \left( \sum b_i \frac{\partial u}{\partial x_i} \right)^2 \]
\[ \sum a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \geq -\max_{1 \leq i \leq n} \int \left( \sum a^{ij}(x) \frac{\partial u}{\partial x_i} \right) u \, dx \]
\[ \geq -\max_{1 \leq i \leq n} \| u \|_{L^2} \left( \sum \| a^{ij} \|^2 + \frac{1}{4 \varepsilon} \int \| u \|_{L^2}^2 \right) \]

where we used

\[ |a_1 \cdot b_1| = 2 \sqrt{\varepsilon} |a_1| \cdot |b_1| \leq 2 \varepsilon |a_1|^2 + \frac{|b_1|^2}{2 \varepsilon} = \frac{\| a_1 \|^2}{2 \varepsilon} \]
Similarly:

\[ \int \left( \sum_i \left( \frac{\partial a_i u}{\partial x^i} \right) \frac{\partial u}{\partial x^i} \right) \, dx \geq \]

\[ - \max_{i} \| \frac{\partial a_i u}{\partial x^i} \|_{\infty} \left( \max_{i} \| \frac{\partial u}{\partial x^i} \|_{\infty} \right) \sum_i \left( \| u \|^2 + \frac{1}{4} \| u \|_{L^2}^2 \right) \]

Finally,

\[ \int c(x) u^2 \, dx \geq -\| c \|_{L^\infty} \| u \|_{L^2}^2 \]

Choose \( \varepsilon \) such that

\[ \varepsilon \left( \max_{i} \| \frac{\partial a_i u}{\partial x^i} \|_{L^2} + \max_{i} \| \frac{\partial u}{\partial x^i} \|_{L^2} \right) \leq \frac{\Theta}{2} \]

then by adding up we get,

\[ B[u,v] \geq \frac{\Theta}{2} \| u \|_{L^2}^2 \]

\[ \quad - \left( \max_{i} \| \frac{\partial a_i u}{\partial x^i} \|_{L^2} + \max_{i} \| \frac{\partial u}{\partial x^i} \|_{L^2} \right) \| u \|_{L^2}^2 \]

\[ \leq \varepsilon \]

Hence

\[ B[u,u] + \gamma \| u \|_{L^2}^2 \geq \frac{\Theta}{2} \| u \|_{L^2}^2 \geq \frac{\Theta}{2C} \| u \|_{L^2}^2 \]
where we used Cauchy inequality (for bounded $U$) in the last inequality.  
\[ \text{Q.E.D.} \]

**Theorem.** For $\nu$ large enough, the problem
\[
\begin{cases}
\sum_{i,j} a^{ij} \partial_i \partial_j u + \sum_i b^i \partial_i u + c(x) u + \mu u = f \\
\partial_i u = 0 \\
|u| = 0
\end{cases}
\]

has a unique weak solution in $H_0^1$ which depends continuously on $f$ provided:

\[ f \in H^{-1}(U), \ b^i, c \in L^\infty(U), \ a^{ij} \in W^{1,\infty}(U), \]

and uniform ellipticity (iii) holds!

**Proof.** Immediate from the two lemmas and Banach-Steinhaus Theorem.

**Theorem (Fredholm Alternative).** For $f \in H^{-1}(U), \ b^i, c \in L^\infty(U), \ a^{ij} \in W^{1,\infty}(U)$ and (iii) holds

then either (ii) has a unique weak solution or...
(adj) \[ B^* [w, v] = B [v, w] = 0 \quad \forall v \in H_0^1 \]

has at least one continuous solution. In this latter case, \((\ast)\) is weakly solvable iff

\[ \langle f, v \rangle = 0 \quad \text{for all } v \in H_0^1 \text{ satisfying } (adj) \]

**Proof.** For the given coeff \(a, b, c\), consider \(y\) given by elliptic estimate lemma and

\[ B [u, v] = B [u, v] + y (u, v) \]

Lax–Milgram Theorem applies to \(B_y\) and

\[ y \in H^{-1} (U) \quad \exists \! u \quad \text{such that} \quad L^{-1} u = y \quad \forall v \in H_0^1. \]

\[ B [u, v] = \langle y, u \rangle \quad \forall u \in H_0^1. \]

Moreover \( L^{-1} : H^{-1} (U) \to H_0^1 (U) \) is linear continuous (from continuous injective in Lax–Milgram) and injective.

Since

\[ H_0^1 (U) \xrightarrow{\text{compact}} H^{-1} (U) \]

then \( L^{-1} : H^{-1} (U) \to H^{-1} (U) \) is compact.
\[ B[u, v] = \langle f, v \rangle \quad \forall u, v \in H_0 \]

\[ B[y, u, v] = \langle yu + f, v \rangle \quad \forall u, v \in H_0 \]

\[ U = L_y^{-1} [yu + f] \]

\[ U - kU = L_y^{-1} f \]

Where

\[ k = yL_y^{-1} : H^{-1}(0) \rightarrow H^{-1}(0) \text{ is compact.} \]

By Fredholm Alternative for compact operators:

\[ (*) \Rightarrow \text{either uniquely solvable } (1 + \beta_p (k)) \]

\[ \exists \ker (\text{Id} - k^{*}) \neq \{0\} \text{ and } (*) \text{ is solvable iff} \]

\[ \langle L_y^{-1} f, v \rangle = 0 \quad \forall v \in \ker (\text{Id} - k^{*}) \]

\[ \forall v \in \ker (\text{Id} - k^{*}) \Rightarrow v = k^{*}u \Rightarrow \]

\[ \langle g, v \rangle = \langle g, k^{*}u \rangle \quad \forall g \in H^{-1}(0) \Rightarrow \]

\[ \langle g, v \rangle = \langle kg, u \rangle \quad \forall g \in H^{-1}(0) \]

But \[ \langle kg, u \rangle = \langle yL_y^{-1} g, u \rangle = (yL_y^{-1} g, u) \]
and from \( B \{ w, v \} = B_y \{ w, v \} - y \{ w, v \} \)
we get
\[
B \{ L^{-1}_y g, v \} = B_y \{ L^{-1}_y g, v \} - (y L^{-1}_y g, v)\
\]
\[
(y L^{-1}_y g, v) = \langle g, v \rangle - B \{ L^{-1}_y g, v \}\
\]
Repeating above we get \( v \in \text{Ker}(I - k^\ast) \Rightarrow \)
\[
B \{ L^{-1}_y g, v \} = 0 \quad \forall g \in H^{-1}(U) \]
but since \( L^{-1}_y \) is onto \( H_0^1(U) \) (surjective) we get
\[
v \in \text{Ker}(I - k^\ast) = B^\ast \{ u, w \} = B \{ w, u \} = 0\
\]
\[
\forall w \in H_0^1(U)\
\]
i.e. \( v \) is a slit of the adjoint \( y \) (adj).
Moreover
\[
\langle L^{-1}_y f, v \rangle = 0\
\]
\[
\Rightarrow \langle y L^{-1}_y f, v \rangle = 0\
\]
\[
\Rightarrow \langle k f, v \rangle = 0\
\]
\[
\Rightarrow \langle f, k^\ast v \rangle = 0\
\]
\[
\Rightarrow \langle f, v \rangle = 0 \quad \text{on} \quad v = k^\ast u. \quad \text{QED}
\]
Remark 2. The proof also shows that when \( \ker (\text{Id} - k^*) = 0 \) then (by composition)
\[
\dim \ker (\text{Id} - k^*) = \dim \ker (\text{Id} - k^*) = \dim H.
\]

But \( U - k(U) = 0 \) \( \iff \langle U, v \rangle = 0 \) \( \forall v \in H \).
So if it solvable the eq has finitely many
linear indep soln's.

Remark 3. \( B^* \left[ U, v \right] = B \left[ U, v \right] \) is the bilinear form induced by the "formal" adjoint
equation:
\[
\begin{cases}
\sum_i \frac{\partial}{\partial x^i} \left( u_i \partial_i v \right) - \sum_i \frac{\partial}{\partial x^i} \left( u^i v \right) + c(x) v = 0 \\
\left. v \right|_{\partial U} = 0
\end{cases}
\]

Theorem (Helmholtz Eq.) There exist a countable
set \( \Sigma = \{ y_1, y_2, \ldots, y_n, \ldots \} \subseteq \mathbb{R} \), with \( n \to \infty \) if
\( \Sigma \) is not finite, such that for \( \lambda \notin \Sigma \) the problem:
\[
\begin{cases}
- \sum_{i,j=1}^{2n} \alpha_i \frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_i b^i \frac{\partial v}{\partial x^i} + c(x) v = \lambda v + f \\
\left. v \right|_{\partial U} = 0
\end{cases}
\]
has a unique weak slu.

Proof. As in the proof of previous Theorem, $\nu$ is a weak slu iff:

$$\nu - (\gamma + \nu) L^{-1} u = L^{-1} f$$

and this is uniquely solvable iff

$$\frac{1}{\gamma + \nu} \notin \sigma(L^{-1})$$

Since $\frac{1}{\gamma + \nu}$ and $L^{-1}$ is compact, this is again:

$$\frac{1}{\gamma + \nu} \notin \sigma(L^{-1}) = \{ \lambda_1, \lambda_2, \ldots, \lambda_k, \ldots \}$$

which

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k| \xrightarrow{k \to \infty} 0$$

if the set is infinite.

$$\Rightarrow \lambda \notin \left\{ \frac{1}{\mu_1 - \gamma}, \frac{1}{\mu_2 - \gamma}, \ldots \right\}$$

$$\lambda_1, \lambda_2, \ldots \text{ and } \lambda_k \to a$$

if the set is infinite.