Summary

4. Applications involving the Laplacian

4.1. Poisson Eq. (elliptic)

4.2. Heat Eq. (parabolic)

4.3. Wave Eq. (hyperbolic)

4.1. Poisson Eq. (elliptic)

\[ -\Delta u = f \quad \text{on } U \subseteq \mathbb{R}^n \text{ open, bounded} \]
\[ u|_{\partial U} = 0 \]

Weak solns: If \( u \in C(\overline{U}) \cap C^2(U) \) solves (x) then:

\[ \int_U \nabla u \cdot \nabla \varphi \, dx = \int_U f(x) \varphi(x) \, dx \quad \forall \varphi \in \mathcal{C}_c^\infty(U) \]
However, the expression above makes sense for \( u \in H^1(U) = W^{1,2}(U) \) and by the density of \( C^\infty(U) \hookrightarrow H^1_0(U) = W^{1,2}_0(U) \) and continuity of the expression above in \( U \) for \( u \in H^1(U) \), \( f \in H^{-1}(U) = W^{-1,2}_0(U) \) we get

\[
(\ast) \quad \int_U \nabla u \cdot \nabla v \, dx = \int_U f \cdot v \, dx \quad \forall v \in H^1_0(U)
\]

If \( \partial U \) is piecewise \( C^1 \) then trace operator

\[ T : H^1(U) \rightarrow L^2(\partial U) \] exists and

\[ u \bigg|_{\partial U} = 0 \quad \text{means} \quad Tu = 0 \quad \Rightarrow \quad u \in H^1_0(U) \]

If \( \partial U \) is not piecewise \( C^1 \) then we can just deal with the case \( u \in L^2(U) \). We define \( u \) is a weak solution of \( (\ast) \) i.f.

\[ u \in H^1_0(U) \text{ and satisfies } (\ast) \text{ for all } v \in H^1_0(U). \]
Theorem (existence & uniqueness of weak sol)

If \( f \in H^{-1}(U) \) then \((**)\) has a unique


Proof: For \( f \in H^{-1}(U) \)

\[
\int_U f \nabla v \, dx = \langle f, v \rangle
\]

is a

continuous linear functional on \( H^1_0(U) \).

If \( \int_U \nabla w \cdot \nabla v \, dx \) were a scalar product

on \( H^1_0(U) \) with respect to which \( H^1_0(U) \) is

Hilbert, then, by Riesz Representation Theorem

there would be a unique \( v \in H^1_0(U) \) such

\[
\int_U \nabla v \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall \, v \in H^1_0(U).
\]

It remains to show

1. \( \int_U \nabla w \cdot \nabla v \, dx \) is a scalar product on

\[ H^1_0(U) \]

2. \( H^1_0(U) \) is Hilbert w.r.t. scalar product

in 1.
1° $\int_0^1 \nabla u \cdot \nabla v \, dx \geq 0$ obvious.

$\int_0^1 \nabla u \cdot \nabla v \, dx = 0 \Rightarrow u = 0$ a.e. needs

the following uniform ellipticity: $\exists C > 0$

$\int_0^1 \nabla u \cdot \nabla v \, dx \geq C \| u \|_{H^1_0}^2$

Recall that for $v \in H^1_0(U)$, $U \subseteq \mathbb{R}^n$ open, $2 < n$

we have the Poincaré inequality:

$\| u \|_{L^2(U)} \leq C_1 \| \nabla u \|_{L^2(U)}$

Since $U$ is bounded we also have:

$\| u \|_{L^2(U)} \leq C_2 \| u \|_{L^{2/(n-2)}(U)}$

Hence $\| u \|_{L^2(U)} \leq C_3 \| \nabla u \|_{L^2(U)}$ for $2 < n$

For $n = 2$ we first use, since $U$ is bounded

$\| \nabla u \|_{L^1(U)} \leq C_4 \| \nabla u \|_{L^2(U)}$
Then, since \( 1 < n \) we can apply Riesz:

\[
\|u\|_{L^{2}(U)} \leq C_2 \|\sqrt{u}\|_{L^{1}(U)}
\]

all in all \( \|u\|_{L^{2}(U)} \leq C_3 \|\sqrt{u}\|_{L^{2}(U)} \) \( \forall u \in H^1_0(U) \)

If \( n = 1 \) we prove by hand:

\[(1) \quad \sup_{x \in U} |u(x)| \leq \|Dv\|_{L^1(U)} \quad \forall v \in C_c(U)
\]

Indeed for \( x \in U \) let \( x_0 \in \partial U \) be the closest boundary point to \( x \) from the segment \( x_0, x \in U \) and

\[
\nu(x) = u(x_0) + \int_{x_0}^{x} \delta(x - s) \, ds
\]

\[
\Rightarrow |u(x)| \leq \int_{x_0}^{x} |u(x)| \, ds \leq \|Dv\|_{L^1(U)}
\]

Taking sup over \( x \in \bar{U} \) gives (1). By density of \( C_c(U) \) in \( H^1_0(U) \) (1) implies:

\[
\|u\|_{L^2(U)} \leq \|\sqrt{u}\|_{L^2(U)} \quad \forall u \in H^1_0(U)
\]

Since \( U \) is bounded we also have
\[ \| \nabla u \|_{L^2(\Omega)} \leq (\text{meas } \Omega)^{1/2} \| \nabla u \|_{L^2(\Omega)} \]

and \[ \| \nabla u \|_{L^2(\Omega)} \leq (\text{meas } \Omega)^{1/2} \| \nabla u \|_{L^2(\Omega)} \]

all \( \Omega \) and \( \| \nabla u \|_{L^2(\Omega)} \leq (\text{meas } \Omega)^{1/2} \| \nabla u \|_{L^2(\Omega)} \)

\( \forall \Omega \) and \( u = 1 \) and

\[ \| \nabla u \|_{L^2(\Omega)} \leq C(u) \| \nabla u \|_{L^2(\Omega)} \]

\( \forall \Omega \) and \( u \in H^1_0(\Omega) \) and \( u \).

\[ \text{W} \text{awe} \quad \| \nabla u \|_{H^1_0(\Omega)} = \| \nabla u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)} \]

\[ \leq C_3 \| \nabla u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)} \]

\[ \leq C_4 \| \nabla u \|_{L^2(\Omega)} , \quad C_4 = C_3 + 1 \]

Hence \( C \| \nabla u \|_{H^1_0(\Omega)}^2 \leq \int \nabla u \cdot \nabla u \, dx \) where

\[ C_4 = 1/C_4 > 0 \]

The particular \( \int \nabla u \cdot \nabla u \, dx = 0 \Rightarrow \)

\[ \| \nabla u \|_{H^1_0(\Omega)} = 0 \Rightarrow u = 0 \text{ a.e.} \]
The symmetry and linearity of \( \int_{\Omega} \nabla w \cdot \nabla v \, dx \) for \( w \) and \( v \) are immediate.

So \( \int_{\Omega} \nabla w \cdot \nabla v \, dx \) def. \( (w, v)_{H_0^1(\Omega)} \) is a scalar product on \( H_0^1(\Omega) \).

2° To show \( (H_0^1(\Omega), (\cdot, \cdot)_L) \) is Hilbert, we use both the ellipticity estimate

\[
(2) \quad C \|v\|^2_{H_0^1(\Omega)} \leq (v, v)_L = \int_{\Omega} \nabla v \cdot \nabla v \, dx
\]

and the following energy estimate:

\[
(3) \quad (v, v)_L = \int_{\Omega} \nabla v \cdot \nabla v \, dx \leq \|v\|^2_{H_0^1(\Omega)}
\]

(2) Shows that any Cauchy sequence \( \{v_n\} \subset H_0^1(\Omega) \)

w.r.t. \( (\cdot, \cdot)_L \) is Cauchy w.r.t. \( H_0^1(\Omega) \). Since \( (H_0^1(\Omega), (\cdot, \cdot)_{H_0^1(\Omega)}) \) is Hilbert, \( \exists v_0 \in H_0^1(\Omega) \) such that \( \|v_n - v_0\|_{H_0^1(\Omega)} \to 0 \).

Using (3) with \( v_n - v_0 \) we get \( \|v_n - v_0\| = \int_{\Omega} \nabla v \cdot \nabla v \, dx \to 0 \).
In conclusion, \((H^0_0(\Omega), (\cdot, \cdot)_1)\) is a Hilbert space. Since \(<f, \cdot>_0\), \(f \in H^{-1}(\Omega)\) is linear, bounded on \((H^0_0(\Omega), (\cdot, \cdot)_1)\):

\[
|<f, \nu>| \leq \|f\|_{H^{-1}(\Omega)} \|\nu\|_{H^0_0(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \frac{1}{\sqrt{c}} \|\nu\|_{H^0_0(\Omega)} \text{ by (2)}.
\]

Hence \(<f, \cdot>_0\) is linear and bounded by \(\|f\|_{H^{-1}(\Omega)} \frac{1}{\sqrt{c}}\) on \((H^0_0(\Omega), (\cdot, \cdot)_1)\).

By Riesz Representation Theorem there exists \(\exists ! \, \nu \in H^0_0\) such that:

\[
<\nu, \varphi> = <f, \varphi> \quad \forall \varphi \in H^0_0(\Omega)
\]

and \(\|\nu\|_{H^0_0(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \frac{1}{\sqrt{c}}\).

Hence (2) has a unique solution \(\nu \in H^0_0(\Omega)\) and by using again (2) we get continuous dependence on \(f\):

\[
(4) \quad \|\nu\|_{H^0_0(\Omega)} \leq \frac{1}{c} \|f\|_{H^{-1}(\Omega)} \quad \text{Q.E.D.}
\]
Remark. The theorem defines:

\[ K : H^{-1}(U) \rightarrow H^1(U) \]

by \( K f = u = \text{unique slab of } (\ast \ast) \).

\( K \) is bijective because for any \( u \in H^1(U) \), \((\langle u, \cdot \rangle) \) defines a unique continuous, linear functional on \( H^{-1}(U) \)

\[ \Rightarrow \exists! f \in H^{-1}(U) \neq k \text{ s.t. } (\langle u, \cdot \rangle) = \text{iff } H^{-1}: K f = u. \]

Then \( K^{-1} : H^1 \rightarrow H^{-1}(U) \) can be viewed as \(-K : H^1 \rightarrow H^{-1}(U)\)

Theorem. \( K \) is linear, bounded, bijective, and:

\[ i_{H^{-1}} \circ K : H^{-1}(U) \rightarrow H^{-1}(U) \]

\[ i_{L^2} \circ K : L^2(U) \rightarrow L^2(U) \]

are both compact, where \( H^1(U) \hookrightarrow H^{-1}(U) \)

and \( H^1(U) \hookrightarrow L^2(U) \) are the usual embeddings.

Proof. Linearity is immediate from the linearity in \( u \) of the LHS of (\( \ast \ast \)) and linearity in \( f \) of RHS of (\( \ast \ast \)).

Boundedness is given by (4) on previous page.

Compactness follows from \( i_{H^{-1}} \) and \( i_{L^2} \)

being compact Q.E.D.
**Theorem** \( \widetilde{K} = \mathcal{L}_2 \circ K |_{L^2(U)} \) is symmetric.

**Proof**

Let \( f, g \in L^2(U) \) and denote

\[
\omega = \langle f, g \rangle, \quad \nu = \langle g, f \rangle
\]

then \( \omega, \nu \in H_0^1(U) \)

and

\[
\int_U \nabla \omega \cdot \nabla \nu \, dx = \int_U f \nu \, dx = \int_U \omega f \, dx
\]

\[
\int_U \nabla \nu \cdot \nabla \nu \, dx = \int_U g \nu \, dx.
\]

Hence \( (g, \langle f, g \rangle) = (Kg, \nu) \forall f, g \in L^2(U) \).

Of course \( Kf = \widetilde{K}f \forall f \in L^2(U) \) since \( K \) is the identity, hence

\[
(g, \langle f, g \rangle) = (\widetilde{K}g, f) \forall f, g \in L^2(U) \text{ Q.E.D.}
\]

Remark 1° The above proof also shows that

\[
(f, \widetilde{K}f) = (f, Kf) = (\nabla \nu, \nabla \nu) \geq 0 \forall f \in L^2(U)
\]

where \( Kf = \nu \).

2° By the properties of linear, symmetric, compact operators, \( \widetilde{K} \) has e-values \( 1, 2, 3, \ldots r \) with corresponding e-vectors dense in \( L^2(U) \).
Corollary (The e-values and e-vectors of explosion)

The problem:

\[- \triangle u = \lambda u \quad \forall \lambda \in \mathbb{R} \]
\[ u |_{\partial U} = 0 \]

Assume a weak s.t. for only a countable \# of \( \lambda \)'s:

\[ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \]

and the corresponding solutions \( \{ u_1, u_2, \ldots , f \} \in H^1_0(U) \)

can be chosen orthonormal in \( L^2 \) and are dense in \( L^2(U), H^1_0(U) \) and \( H^{-1}(U) \) with:

\[ f = \sum_{k=1}^{\infty} \langle f, u_k \rangle u_k \quad \forall \ f \in L^2(U) \text{ or } f \in H^1_0(U) \]

\[ f = \sum_{k=1}^{\infty} \langle f, u_k \rangle u_k \quad \forall \ f \in H^{-1}(U) \]

the convergence of the sum on \( \text{RHS} \) is in norm in \( L^2(U) \) respectively the norm in \( H^1_0(U) \) and \( H^{-1}(U) \).

Proof: weak s.t.'s of (\( \ast \)) are given

\[ u = K(\lambda u) \]

Since \( K \) (linear \( \Rightarrow K0 = 0 \Rightarrow \lambda = 0 \) cannot admit a nontrivial weak s.t.). Then the eq is:

\[ (K - \frac{1}{\lambda} I) u = 0 \]
Using Remark 2° above we set:

\[ A_k = \frac{1}{\nu_k} \Rightarrow 0 < \nu_k < 12 \leq \ldots \]

\[ v_k = f_k \quad \text{where} \quad \{f_k\}_{k=1} \text{is the} \]

orthonormal set of e-vectors for \( K \) on \( L^2(U) \).

Note that

\[ \frac{1}{\nu_k} f_k = f_k \Rightarrow v_k = f_k \in H_0^1(U) \]

Since \( \{v_k\}_{k=1} \) is dense in \( \text{Range} \ K \) and

\[ L^2(U) = \ker K \oplus \text{Range} \ K \]

we get \( \{v_k\}_{k=1} \) is dense in \( L^2(U) \) and since

they are orthonormal:

\[ f = \sum_{k=1}^{\infty} (v_k, f) v_k \quad \forall f \in L^2(U) \]

For \( f \in H_0^1(U) \subseteq L^2 \), the formula stands but we need to show convergence in \( H_0^1 \) norm.

Note that \( (\nabla v_k, \nabla v_j) = \lambda_k (v_k, v_j) = \lambda_k \delta_{kj} \)

Hence \( \{v_k\}_{k=1} \) are orthonormal in \( H_0^1 \) under the "new"
scalar product \( \langle \nabla u, \nabla v \rangle \). It is also complete because \( \langle \nabla u_k, \nabla v \rangle = 0 \) \( \forall k \in \mathbb{N} \Rightarrow \Lambda_k (u_k, v) = 0 \Rightarrow \langle u_k, v \rangle = 0 \) \( \forall k \in \mathbb{N} \Rightarrow v = 0 \).

Hence, for \( f \in H^1_0 (U) \):

\[
\frac{f}{V} = \sum \frac{\langle \nabla u_k , \nabla f \rangle}{\sqrt{\Lambda_k}} \frac{u_k}{\sqrt{\Lambda_k}} = \sum \frac{\Lambda_k (u_k, f) u_k}{\sqrt{\Lambda_k} \sqrt{\Lambda_k}} = \sum (u_k, f) u_k \text{ converging in particular that the sum converges in } \| \cdot \|_V \text{ and } H^1_0 \text{ which being equivalent to the usual norm } \Rightarrow \text{ the convergence is also in the norm of } H^1_0.

For \( f \in H^{-1} (U) \) we consider \( Kf \in H^1_0 (U) \).

Recall \( K: H^{-1} (U) \to H^1_0 (U) \) is linear, continuous and bijective, so \( K^{-1} (f) = -f \) also, is linear and continuous (by the open mapping theorem)

By the previous argument we have:

\[
Kf = \sum (Kf, u_k) u_k
\]

\[
= K^{-1} Kf = K^{-1} \left[ \sum (Kf, u_k) u_k \right]. \text{ Since the sum is convergent in } H^1_0 \text{ norm and } K^{-1} \text{ is continuous, we get}
\]
\[ f = \sum_{k=1}^{\infty} \langle k, \psi_k \rangle \psi_k \] with the same convergence in the norm of \( H^{-1}(U) \). Now \( K^{-1}\psi_k = \phi_k \psi_k \) and \( \langle k, \phi_k \psi_k \rangle = \langle \Delta k, \Delta \psi_k \rangle = \langle f, \psi_k \rangle \) for all \( k \).

Thus in all
\[ f = \sum_{k=1}^{\infty} \langle k, \psi_k \rangle K^{-1}\psi_k = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k \] Q.E.D.

**Corollary** (Diagonalization of \(-\Delta\)). The weak solution of
\[ \begin{cases} -\Delta u = f & \text{in } H^{-1}(U) \\ u|_{\partial U} = 0 \end{cases} \]

is given by
\[ u = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k \]

**Proof** \( u = Kf \) and by spectral theorem for linear, symmetric, compact operators: if \( f \in L^2(U) \):
\[ Kf = \sum_{k=1}^{\infty} \langle \psi_k, f \rangle \psi_k = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle \psi_k, f \rangle \psi_k \]

If \( f \in H^{-1}(U) \) then \( f = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k \) and since the sum is convergent in norm \( \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \) and \( K \) is continuous:
\[ Kf = K \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k \]
4.2. Heat Eq. (Parabolic)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\Delta u & t > 0 \\
\frac{\partial u}{\partial t} &= 0 \\
u(0) &= u_0
\end{align*}
\]

**Def:** A **strong** **solution** of (2.1) is a function \( u : \mathbb{R} \times [0,T] \to H_0^1(U) \) continuous, with \( u \in C^1(\mathbb{R} \times [0,T], H^{-1}(U)) \cap C(\mathbb{R} \times [0,T], H^1(U)) \) such that

\[
\langle \frac{\partial u}{\partial t}, u \rangle = -\langle \nabla u, \nabla u \rangle \quad \text{for all } t \geq 0, \quad u \in H_0^1(U)
\]

and \( u(0) = u_0 \) in \( H_0^1(U) \).

**Theorem** \( \forall u_0 \in H_0^1(U) \) there exist a unique **strong solution** of (2.1)

**Proof** Let \( \{(\lambda_k, u_k)\}_{k \in \mathbb{N}} \) be the \( \lambda \)-values and \( u \)-vectors of \(-A\). Recall \( u_k \in H_0^1(U) \) and any \( f \in L^2(U) \) can be written:

\[
f = \sum_{k=1}^{\infty} \langle f, u_k \rangle u_k
\]
Any strongly stationary $u(t) \in H^1_0(U) \subset L^2(U)$ for $0 \leq t < T$. Hence:

$$u(t) = \sum_{k=1}^{\infty} d_k(t) u_k \quad d_k = \langle u, u_k \rangle$$

Since $\frac{\partial u}{\partial t} \in C((0,T), H^{-1}(U))$, we get

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} w_k(t) u_k$$

where $w_k(t) = \langle \frac{\partial u}{\partial t}, u_k \rangle = \frac{d}{dt} \langle u, u_k \rangle = -\sum_{k=1}^{\infty} \lambda_k d_k(t) \langle u_k, u \rangle$

Then (1.1) becomes

$$\sum_{k=1}^{\infty} \lambda_k d_k(t) \langle u_k, v_k \rangle = -\sum_{k=1}^{\infty} \lambda_k d_k(t) \langle u_k, v \rangle$$

Using $v = u_j, j = 1, 2, \ldots$ we get

$$d_j'(t) = -\lambda_j \cdot d_j$$

while from $u(0) = u_0$ we get:

$$d_j(0) = \langle u_0, u_j \rangle$$
Consequently
\[ \phi_j(t) = e^{-\lambda_j t} (u_0, u_j) \]
and the unique solution would be (if we show convergence)
\[ u(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} (u_0, u_j) u_j \]

The convergence of this series in $H_0^1(U)$ is immediate from $u_0 \in H_0^1(U)$, the continuity of $e^{-\lambda_j t}$ implies $u(t) \in C([0, T), H_0^1(U))$.
Moreover, \( \frac{\partial u}{\partial t} \in C([0, T), H^{-1}(U)) \) follows from
\[ \sum_{j=1}^{\infty} \frac{\partial}{\partial t} e^{-\lambda_j t} (u_0, u_j) u_j \text{ convergent in } H^{-1} \]
and
\[ K \left[ \sum_{j=1}^{N} e^{-\lambda_j t} (u_0, u_j) u_j \right] = \sum_{j=1}^{N} \frac{\partial}{\partial t} \left( \frac{\nabla u_0}{\nabla u_j} \frac{\nabla u_j}{\nabla u_j} \right) u_j \]

and the latter is convergent in norm near $H_0^1$, since
\[ \left| e^{-\lambda_j t} \left( \frac{\nabla u_0}{\nabla u_j} \frac{\nabla u_j}{\nabla u_j} \right) \right|^2 \leq \left| \left( \frac{\nabla u_0}{\nabla u_j} \frac{\nabla u_j}{\nabla u_j} \right) \right|^2. \]