Summary:

3. Linear maps between normed spaces

3.4 Compact operators and their spectrum (cont.)

3.4. Compact operators and their spectrum (continuation)

Let $X$ be a normed space over the field $K = \mathbb{R}$ or $\mathbb{C}$.

**Theorem** If $A : X \to X$ is compact then

(i) Range $(A - \text{Id})$ is closed.

(ii) $\ker (A - \text{Id})$ is finite dimensional and has the same dimension as $\ker (A^* - \text{Id})$.

(iii) If $\lambda \in \sigma(A)$ then either $\lambda \in \sigma(A)$ and has finite multiplicity or $\lambda \in \sigma^p(A)$.

(iv) $\sigma(A) \subseteq \{0\} \cup \sigma^p(A)$ and $0$ may be the only accumulation point of the spectrum.
Proof of Theorem:

For part (ii), finite dimensionality part:
Assume $\dim \ker (A - Id) > 0$. Choose 
$\{x_n\}_{n \in \mathbb{N}} \subseteq \ker (A - Id)$ linearly independent such that $\|x_n\| = 1$ and 
$$\text{dist}(x_n, \text{span}\{x_1, x_2, \ldots, x_n\}) > \frac{1}{2}.$$ 
Then $\{x_n\}$ is bounded $\Rightarrow A(x_n)$ has a convergent subsequence $A(x_{n_k})$. But $A(x_{n_k}) = x_{n_k} \Rightarrow x_{n_k}$ is convergent and (i) shows that $\{x_n\}$ cannot be Cauchy. 

For part (i) we use the following lemma:

Lemma 4. There $\exists x > 0$ such that 
$$\forall y \in \text{Range} (A - Id), \exists x \in X \text{ such that } \|x\| \leq 2\|y\| \text{ and } Ax - x = y.$$

Proof of Lemma. If $X$ were Banach and $A - Id$ bijective then this would be a consequence of the open mapping theorem. Under the hypotheses of the lemma we do the proof by contradiction.
Assume \( A \in M \exists y_n \in \text{Rouge} \quad (A - I_d), y_n \to 0 \) such that for all \( x \in X \) satisfying \( Ax - x = y_n \),
\[
\|x_u - x\| = \inf \{\|x_u - z\| : z \in \text{ker} \quad (A - I_d)\} = \|x_u - z\|
\]

Furthermore
\[
\{ x \in X : Ax - x = y_n \} = \{ x_n - z : A x_n - x_n = y_n \quad \text{and} \quad z \in \text{ker} \quad (A - I_d) \}
\]

Since \( \text{ker} \quad (A - I_d) \) is closed (valid for any continuous op) and finite dimensional then
\[
\exists z_n \in \text{ker} \quad (A - I_d) \text{ such that}
\]
\[
\inf \{\|x_n - z\| : z \in \text{ker} \quad (A - I_d)\} = \|x_n - z\|
\]

Let \( x_n = x_u - z_n, u_n = \frac{x_n}{\|x_n\|}, v_n = \frac{y_n}{\|x_n\|} \)

Then \( A(x_n) - x_n = y_n \Rightarrow \|x_n\| \geq \|y_n\| \)

\[
\Rightarrow \|u_n\| \leq \frac{1}{\|v_n\|} \Rightarrow v_n \to 0
\]

\( \{v_n\} \) bounded by 1. \( A \) compact \( \Rightarrow \exists v_n \) such that \( A(v_n) \) is convergent to \( v \).

But
\[
\lim v_n = A(v_n) = v_n + u_n
\]

\( \downarrow 0 \)
Hence $U_{n_k} \to U$ and $A(U_{n_k}) \to \frac{1}{2} A(u)$

$\Rightarrow A(u) = u \Rightarrow u \in \ker (A - Id).$

$$\|U_{n_k} - U\| = \|x_{n_k} - U\| \leq \frac{1}{2} \|x_{n_k} - U\|$$

$$\|x_{n_k} - U\| \leq \frac{1}{2} \|x_{n_k} - u\|$$

$$\xrightarrow{\ker A - Id}$$

Hence $U_{n_k} \to U$ contradiction.

Return to part (i) i.e. $\ker (A - Id)$ is closed. Let $u \in \ker (A - Id)$ such that $y_n \to y \in X$. By Lemma 1, $\exists x > 0$ such that for any $u$,

$$y_n = A x_{n_k} - x_n$$

and $\|x_{n_k} - u\| \leq \|y_n\|$

Since $\{y_n\}$ convergent $\Rightarrow \frac{1}{2} y_n$ convergent $\Rightarrow \frac{1}{2} y_n$ bounded.

$\Rightarrow \exists \{x_{n_k}\}$ such that $A(x_{n_k}) \to x$ compact

Then $x_{n_k} - y_{n_k} \to x - y$.

$A$ continuous implying $x = \lim_{k \to \infty} A(x_{n_k}) = A(x - y)$.
Equivalently \( y = A(x-y) - (x-y) \) or
\[ y = (A - \text{Id})(x-y) \in \text{Ker}(A - \text{Id}) \).

Part (iii) Consider first \( N = 1 \).
If \( 1 \in \text{supp} \phi(A) \) we need to show that
\[ \dim \left( \text{span} \left\{ x \in X : \exists n \in \mathbb{N} \ (A - \text{Id})^n(x) = 0 \right\} \right) = \infty. \]

Assume contrary and denote
\[ X_n = \left\{ x \in X : (A - \text{Id})^n(x) = 0 \right\}. \]

Clearly \( X_n \subseteq X_{n+1} \quad \forall n \in \mathbb{N} \). If \( \exists n \in \mathbb{N} \)
such that \( X_n = X_{n+1} \), then
\[ X_n = X_{n+1} \quad \forall n \in \mathbb{N} \]

indeed from \( x \in X_{n+2} \Rightarrow (A - \text{Id})^{n+2}(x) = 0 \)
\[ \Rightarrow (A - \text{Id})^{n+1}(A - \text{Id})(x) = 0 \]
\[ \Rightarrow (A - \text{Id})(x) \in X_{n+1} = X_n \]
\[ \Rightarrow (A - \text{Id})^m(A - \text{Id})(x) = 0 \Rightarrow x \in X_{m+1} = X_m \]
\[ \Rightarrow X_{n+2} \subseteq X_n. \quad \text{Inductively one gets} \]
\[ X_n = X_{n+1} = \ldots = X_{n+m} = \ldots \]
Hence \( \text{span} \left\{ x \in X : \exists n \in \mathbb{N} \ (A - \text{Id})^n(x) = 0 \right\} \neq X_m \)
But \( X_n = \ker (A - I_n)^w = \ker \left( A^w - A^{w-1} + \cdots + (-1)^{w-1} A + (-1)^w \right) \) is compact.

By part (ii) \( \dim X_n < \infty \) contradiction.

Hence \( X_1 \subset X_2 \subset X_3 \subset \cdots \). As in part (ii) we can choose \( x_n \in X_n \) with \( \| x_n \| < 1 \) and \( \text{dist} (x_n, X_{n-1}) > 1/2 \).

Since \( \| x_n \| = 1 \Rightarrow A(x_n) \) has a convergent subsequence.

But for \( u > w \):

\[
\| A(x_u) - A(x_w) \| = \| x_u - x_w + (A - I_u) (x_u - x_w) \|_{X_{u-w}} > \frac{1}{2}
\]

Contradiction.

If \( 1 \notin \sigma_p (A) \) we need to show that \( I \in \sigma (A) \).

We use the following two lemmas.

\[ \text{Lemma 2} \quad y^* \in \text{Rouge} (A^* - I_d) \Rightarrow \ker (A - I_d) \subset \ker (A^* - I_d) \]

\[ \text{Lemma 3} \quad \text{If} \ Z \text{ is a normed space and} \]

\[ B : Z \to Z \text{ is linear and compact then} \ B - I_d \text{ surjective implies} \ B - I_d \text{ injective.} \]
Now, \( 1 \in \Gamma_p(A) \Rightarrow \ker (A-\text{Id}) = \{0\} \) hence \( \ker (A-\text{Id}) \leq \ker y^* \forall y^* \in X^* \Rightarrow A^* - \text{Id} \) is surjective by Lemma 2 \Rightarrow A^* - \text{Id} \) is injective by Lemma 3 \Rightarrow \ker (A^* - \text{Id}) = \{0\} \Rightarrow X = \ \text{Rough}(A-\text{Id})

and since the rough is closed, part (i), use closure \( A-\text{Id} \) bijective \Rightarrow (A-\text{Id})^{-1} \) exists and by Lemma 1 it is continuous

\( \Rightarrow 1 \in \mathcal{R}(\mathcal{A}). \)

If \( 0 \neq \lambda \neq 1 \) then \( A - \lambda \text{Id} = \lambda (\frac{A}{\lambda} - \text{Id}) \) and the argument above for \( \frac{A}{\lambda} - \text{Id} \) with \( \frac{A}{\lambda} \)

compact finishes Part (ii).

We still need to prove Lemmas 2, 3.

Proof of Lemma 2. Observe that for a reflexive Banach space, Lemma 2 is simply the Euclidean alternative.

\[ \implies + \times \]

\[ \leftarrow \] Consider \( x^* : \text{Rough}(A-\text{Id}) \to K \)

by:

\[ x^*(x) = x^*(Ay - y) = y^*(y) \]
If both $y_1, y_2$ satisfy $x = Ay_1 - y_1 = Ay_2 - y_2$,

\[ y_1 - y_2 \in \ker (A - \text{Id}) \Rightarrow y_1 - y_2 \in \ker y^* \]

\[ y^*(y_1) = y^*(y_2) \text{ so } x^* \text{ is well defined, linear and continuous:} \]

\[ |x^*(x)| = |y^*(y)| \leq \|y\|^* \|y\| \]

But $y : x = Ay - y$ can be chosen such that

\[ \|y\| \leq \alpha \|x\| \text{ for some } \alpha > 0 \text{ independent of } x \]

(see Lemma 1) so:

\[ |x^*(x)| \leq \|y\|^* \|x\| \Rightarrow x^* \text{ bounded} \]

Since $\text{Rouge}(A - \text{Id}) = \text{domain of } x^*$ is linear and closed and $x^*$ is linear and continuous,

\[ \Rightarrow \text{ } x^* \text{ can be prolonged to } X \text{ and remains bounded by } \|y\|^* \alpha \].

Now; $\forall y \in X$

\[ \langle (A - \text{Id})^* x^*, y \rangle = \langle x^*, (A - \text{Id})y \rangle \]

\[ \text{def} \ x^* = \langle y^*, y \rangle \forall y \in X \]

\[ \Rightarrow (A - \text{Id}) x^* = y^* \Rightarrow y^* \in \text{Rouge}(A^* - \text{Id}). \]
Proof of Lemma 3: Assume \((B-I_d)\) is not injective but not surjective.

Consider for \(u \in \mathbb{N}\)

\[ X_u = \{ x \in X : (B-I_d)^u(x) = 0 \}. \]

So \(x_i \in X_1\) and by surjectivity \(\exists x_2 \in X\) such that

\[(B-I_d)(x_2) = x_i \neq 0 = c x_2 \in X_1\]

and \((B-I_d)^2(x_2) = (B-I_d)x_i = 0 = c x_2 \in X_2\)

Inductively we can construct \(\{X_n\}_{n \in \mathbb{N}}\) such that \(c x_i = x_i\), \((B-I_d)(x_{n+1}) = x_n\) and consequently \(x_{n+1} \in X_n \smallsetminus X_n \forall n \in \mathbb{N}\)

\(=\) claim \((\text{span} \{ x \in X : \exists u \in \mathbb{N} (B-I_d)^u x = 0 \})\) is not finite. This contradicts the argument in the first part of (iii) unless \(A\) replaced by the compact \(B\).

Part (iv): From Part (iii) we have

\[ 0(A) = \{ 0 \} \cup \sigma_p(A) \]

We will show that for any \(n > 0\)

\[ \sigma_n \overset{\text{def}}{=} \{ \lambda \in 0(A) : \lambda > n \} \text{ is finite} \]
Assume contrary \( \forall \{ \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \} \subseteq \mathbb{T} \) countable. Since \( \mathbb{T} \subseteq \mathcal{E}(A) \) we know:
\[ \{ x_1, x_2, \ldots, x_n, \ldots \} \subseteq X \] for such that
\[ Ax_i = \lambda_i x_i \]

Let \( X_n = \text{Span} \{ x_1, x_2, \ldots, x_n \} \) \( n \in \mathbb{N} \)

Then \( X_n \subseteq X_{n+1} \) and \( X_n \neq X_{n+1} \) because \( x_{n+1} \) corresponds to an \( \epsilon \)-value distinct from \( \lambda_1, \ldots, \lambda_n \).

We can then inductively construct the sequence \( \{ X_n \}_{n=0}^\infty \):

\[ x'_1 = x_1 / \| x_1 \| \]
\[ x'_n \in X_n \quad \| x'_n \| = 1 \quad \text{dist}(x'_n, X_{n-1}) > \frac{1}{2} \]

Let \( x''_n = \frac{1}{\lambda_n} x'_n \) \( n \in \mathbb{N} \)
\( \| x''_n \| = \frac{1}{\lambda_n} \leq \frac{1}{\lambda} \)

Since \( \{ x''_n \}_{n=0}^\infty \) is bounded \( \Rightarrow A(x''_n) \) has a convergent subsequence. But for \( m < n \)

\[ \| A(x''_m) - A(x''_n) \| = \| \frac{1}{\lambda_m} A \left( \sum_{i=1}^m \beta_i x_i \right) - A(x''_n) \| = \]
\[
\| \frac{1}{\lambda_m} \sum_{i=1}^{m} \beta_i \lambda_i x_i + x'_m - \sum_{i=1}^{m} \beta_i x_i - A(x''_m) \| \\
= \| x'_m - \left[ \sum_{i=1}^{m-1} \beta_i (1 - \frac{\lambda_i}{\lambda_m}) x_i + \frac{1}{\lambda_m} \sum_{j=1}^{m} \lambda_j \lambda_j x_j \right] \| \\
\subseteq X_{m-1} \\
\geq \frac{1}{2}
\]

Hence no subsequence of \( A(x_m) \) can be Cauchy.

Contradiction.

It remains to show

\[ \dim \ker(A - I_d) = \dim \ker(A^* - I_d). \]

If \( \ker(A - I_d) = \{0\} \Rightarrow A^* - I_d \text{ surjective} \]
by Lemma 2 \( \Rightarrow \) \( A^* - I_d \) injective by Lemma 3
\[ \Rightarrow \ker(A^* - I_d) = \{0\} \]
Fredholm Alternative

If \( \ker(A^* - I_d) = \{0\} \Rightarrow \text{Range}(A - I_d) = X \]
\[ \Rightarrow A - I_d \text{ is surjective by part (i) } \Rightarrow A - I_d \text{ is } \]

Injective by Lemma 3 \(\Rightarrow\) \(\text{ker}(A - \text{Id}) = \{0\}\).

It remains to show \(w = u\) unless

\[ w = \text{cl}\text{. ker}(A - \text{Id}), \quad u = \text{cl}\text{. ker}(A^\infty - \text{Id}) \]

and one both follows by the first part of (ii), note that \(A^\infty\) is also compact.

Let \(\{e_1, e_2, \ldots, e_m, y\}, \{b_1, b_2, \ldots, b_n\}\) bases in \(\text{ker}(A - \text{Id})\) respectively \(\text{ker}(A^\infty - \text{Id})\). Use Hahn-Banach to construct \(\{x_1, \ldots, x_n, y\} \subset \mathbb{X}\) such that:

\[ x_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad b_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

Assume \(w < u\). Define

\[ B : X \to X, \quad B(x) = \sum_{i=1}^m x_i^e(x) x_i, \quad C = A + B \]

Then \(B\) and \(C\) are linear and compact. Moreover if \((C - \text{Id})(x) = 0\) then:

\[ (A - \text{Id})(x) = -\sum_{i=1}^m x_i^e(x) x_i \]
But \( \langle b_j^*, (A-I_d)(x) \rangle = \langle (A^* - I_d) b_j^* , x \rangle \)
\[
= \langle 0 , x \rangle = 0 \quad \text{since} \quad b_j^* \in \ker A^* - I_d
\]
\[
\implies 0 = b_j^* \left( \sum_{i=1}^{m} x^i(x) x_i \right) = -x_j^*(x)
\]
\[
\implies x_j^*(x) = 0 \quad \forall j = 1, \ldots, m, m+1, \ldots, n = c
\]
\[
\sum_{i=1}^{m} x^i(x) x_i = 0 \implies (A-I_d)(x) = 0 \implies x \in \ker (A-I_d)
\]
\[
\implies \exists d_1, \ldots, d_m \in \mathbb{K} : x = \sum_{i=1}^{m} d_i x_i
\]
\[
\implies 0 = x_j^*(x) = d_j = 0 \quad j = 1, m
\]
\[
\implies x = 0 \implies (C-I_d) \text{ injective} \quad (\text{ii})
\]
\[
1 \in \mathcal{P}(C) \implies (C-I_d) \text{ surjective} \implies \exists x \in X
\]
\[
(C-I_d)(x) = x_{m+1}
\]
\[
\implies 1 = b_{m+1}^*(x_{m+1}) = \langle b_{m+1}^*, (A-I_d)(x) \rangle
\]
\[
+ \sum_{i=1}^{m} x_i^*(x) b_{m+1}^*(x_i) = 0 \quad \text{contradiction}
\]
\[ m \geq n. \] The case \( m > n \) is treated by a similar (dual) argument using:

\[ B^*: X^* \rightarrow X^*, \quad B^*(x^*) = \sum_{i=1}^{n} x^*_i (x_i) x_i^*, \quad C^* = A^* v^* \]

All in all \( m = n \) and the theorem is now finished. Q.E.D.

**Corollary:** The equations:

1. \[ Ax - x = y, \quad y \in X \]
2. \[ A^* x^* - x^* = y^*, \quad y^* \in X^* \]

are either both uniquely solvable or

\[ Ax - x = 0 \]
\[ A^* x^* - x^* = 0 \]

Done the same number of finitely many independent steps.

Proof. \[ \pm 1 x. \]
The case of linear, compact, symmetric operators

Def. $H$ pre-Hilbert space $A : H \to H$ linear is called symmetric if:

$$\langle Au, v \rangle = \langle u, Av \rangle \quad \forall u, v \in H$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product on $H$.

Remarque 1) $H$ Hilbert then $A$ linear and symmetric $\Rightarrow A$ continuous

2) $H$ Hilbert then $A$ linear and symmetric $\Rightarrow A = A^* \quad$ (or $A$ is self-adjoint)

Theorem. $H$ pre-Hilbert space $A : H \to H$ linear, continuous and symmetric then:

(i) $\|A\| = \sup \{ |\langle A(u), u \rangle| : u \in H, \|u\| = 1 \}$

(ii) $\sigma_p(A) \subseteq [-\|A\|, \|A\|] \subset \mathbb{R}$

(iii) If $\lambda_1 + \lambda_2 \in \sigma_p(A)$ and $Au_1 = \lambda_1 u_1$, $Au_2 = \lambda_2 u_2$ then $\langle u_1, u_2 \rangle = 0$
Proof

(i) Since \( \| A \| = \sup \{ \| A(x) \| : x \in H, \| x \| = 1 \} \)

Then \( \langle A(u), v \rangle \leq \| A u \| \| v \| \leq \| A \| \| u \| \| v \| \)

\[ \| A \| \| u \| = 1 \]

\[ \| A \| = \sup \{ \| A(u), v \| : \| u \| = 1 \} \]

Reciprocal use

\[ \| A(u) \| ^2 = \frac{1}{4} \left[ \langle A(\lambda u + \frac{1}{\lambda} A(u)), \lambda u + \frac{1}{\lambda} A(u) \rangle \right. \]

\[ - \langle A(\lambda u - \frac{1}{\lambda} A(u)), \lambda u - \frac{1}{\lambda} A(u) \rangle \right] \]

with \( \lambda = \frac{\| A(u) \|}{\| u \|} \) for \( A u \to 0 \) to show

\[ \| A(u) \| ^2 \leq N_A \| A(u) \| \| u \| \Rightarrow N_A \geq \| A \| \]

(ii) \( \Omega \subset \mathbb{R} \) let \( x \in \Omega \) = 0 \( A u = x u \)

\[ \Rightarrow \langle A u, v \rangle = \langle x u, v \rangle \]

\[ \Rightarrow \langle u, A u \rangle = x \langle u, v \rangle \]

\[ \Rightarrow \langle u, A u \rangle = \bar{x} \langle u, u \rangle \Rightarrow \| u \| ^2 = x \| u \| ^2 \]
\[ n = \overline{n} \quad \Rightarrow \quad n \in \mathbb{R}. \]

If \[ |n| > \|A\| \]
then

\[ \langle (\lambda I_d - A)u, u \rangle \geq (|n| - \|A\|) \|u\|^2 \]

\[ \Rightarrow \]

there is no \( u \neq 0 \) such that \( (\lambda I_d - A)u = 0 \)

(iii) \[ \langle Au_1, u_2 \rangle = \overline{\lambda_1} \langle u_1, u_2 \rangle = \lambda_1 \langle u_1, u_2 \rangle \]
\[ \langle Au_1, u_2 \rangle = \overline{\lambda_1} \langle u_1, Au_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle \]

\[ \Rightarrow \quad 0 = (\lambda_2 - \lambda_1) \langle u_1, u_2 \rangle \quad \Rightarrow \quad \langle u_1, u_2 \rangle = 0 \]

**Theorem** \[ \text{pre-Hilbert } A : H \rightarrow H \text{ linear compact and symmetric then } \exists \lambda \in \sigma_p(A) \]

such that \[ |\lambda| = \|A\| \]

**Proof** \[ \text{if } Au = 0 \quad \forall u \in H \text{ then done.} \]

Otherwise let \[ \{u_n\} \subset H \quad \|u_n\| = 1 \quad \forall n \text{ be such that} \]

\[ |\langle Au_n, u_n \rangle| \rightarrow \|A\| \]

then \[ \exists \{u_{n_k}\} \subset \{u_n\} \text{ such that either} \]

\[ \langle Au_{n_k}, u_{n_k} \rangle \rightarrow \|A\| \]

\[ \text{or} \quad \langle Au_{n_k}, u_{n_k} \rangle \rightarrow -\|A\| \]
Consider \( \langle A u_{n_k}, u_{n_k} \rangle \to 1 \) all the
others is treated similarly.

Let \( n = \|A u\| \). Then
\[
\| A u_{n_k} - n u_{n_k} \| \to 0 \quad k \to \infty.
\]

Indeed
\[
\| A u_{n_k} - n u_{n_k} \|^2 = \langle A u_{n_k} - n u_{n_k}, A u_{n_k} - n u_{n_k} \rangle
\]
\[
= \| A u_{n_k} \|^2 - 2 n \langle A u_{n_k}, u_{n_k} \rangle + n^2 \| u_{n_k} \|^2
\]
\[
\leq \| A \|^2 - 2 n \langle A u_{n_k}, u_{n_k} \rangle + n^2
\]
\[
\to 0 \quad k \to \infty.
\]

But \( u_{n_k} \) bounded \( \Rightarrow \exists A u_{n_k}, \to 0 \)
\[
\Rightarrow u_{n_k} \to 0 \quad \text{and by continuity of } A
\]
\[
(A - n) u = 0 \Rightarrow \lambda \in \sigma_p(A) \quad Q.E.D.
\]

Algorithm for constructing e-values of the
linear, compact, symmetric operator \( A \) on a
pre-Hilbert space:

1. Compute \( \sup \left\{ \left| \langle A u, u \rangle \right| : \|u\| = 1 \right\} = \|A\| \)

2. If \( \|A\| = 0 \) stop: \( \sigma(A) = \sigma_p(A) = 0 \)
If $\|A\| \neq 0$ then $\exists \lambda_1 \in \sigma (A) \\setminus \{0\}$, $\|\lambda_1\| = \|A\|
and \exists u_1, \|u_1\| = 1, Au_1 = \lambda_1 u_1.$

$2^o \quad H_1 = \{u, y^\perp \}.$ Compute

$$\sup \left\{ \langle Au, u \rangle \mid u \in H_1, \|u\| = 1, y = \|A\|u \right\}$$

($A_1 = A |_{H_1}$)

If $\|A_1\| = 0$ stop $\sigma (A) = \{\lambda_1, 0\}.$

else $\exists \lambda_2 \in \sigma (A) \\setminus \{0\}, \|\lambda_2\| = \|A_1\|$
and $u_2 \in H_1, \|u_2\| = 1, Au_2 = \lambda_2 u_2.$

$3^o \quad H_2 = \{u_1, u_2, y^\perp \}.$ Compute

$$\sup \left\{ \langle Au, u \rangle \mid u \in H_2, \|u\| = 1, y = \|A_2\|u \right\}$$

($A_2 = A |_{H_2}$)

If $\|A_2\| = 0$ stop $\sigma (A) = \{\lambda_1, \lambda_2, 0\}.$

else $\exists \lambda_3 \in \sigma (A) \\setminus \{0\}, \|\lambda_3\| = \|A_2\|$
and $u_3 \in H_2, \|u_3\| = 1, Au_3 = \lambda_3 u_3.$
Theorem (i) \(|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \ldots \geq 0\) if \(\lambda_j\)'s are not finite then \(\lambda_j \to 0, j \to \infty\).

(ii) \(\sigma(A) \setminus \{0\} = \{\lambda_1, \lambda_2, \ldots\}\).

(iii) \(A(u) = \sum_{k=1}^{\infty} \lambda_k \langle u_k, u \rangle u_k\) \(\forall u, u_k\) and:

(Spectral representation). If in addition \(H\) is bilinear then:

\[H = \ker(A) \oplus \text{Ran}(A)\]

and if \(H\) is separable \(\Rightarrow\) \(\exists\{v_0, v_1, \ldots\}\) orthogonal basis in \(\ker A\) (\(\Rightarrow v_0\) e-vector for \(n = 0\) and:

\[u = \sum_{k=1}^{\infty} \langle u_0, u \rangle u_k + \sum_{k=1}^{\infty} \langle u_0, u \rangle u_k\]