Convergence Theorems for Lebesgue Integral

Definition (Lebesgue Integral) \( f : \mathbb{R}^n \to \mathbb{R} \) is Lebesgue integrable if there exist \( f^+, f^- \in C_c^0 \) such that \( f = f^+ - f^- \). In this case

\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} f^+(x) \, dx - \int_{\mathbb{R}^n} f^-(x) \, dx
\]

Recall \( C^+ = \{ f : \mathbb{R}^n \to \mathbb{R} \mid \exists g \in L^1(\mathbb{R}^n) \text{ a.e.} \} \)

with \( g \) piecewise continuous and compactly supported \( g \)

\[
C^+_0 = \{ f \in C^+ \mid \int_{\mathbb{R}^n} g_k(x) \, dx \text{ is uniformly bounded} \}
\]

Here \( \int_{\mathbb{R}^n} g_k(x) \, dx \) is a Riemann integral
Remark. For \( f \in C^+ \setminus C^* \) we can define
\[
\int_{\Omega^n} f(x) \, dx = +\infty \quad \text{and we have:}
\]

(i) If \( f, g \in C^+ \) then \( f + g \in C^+ \) and
\[
\int_{\Omega^n} (f + g)(x) \, dx = \int_{\Omega^n} f(x) \, dx + \int_{\Omega^n} g(x) \, dx
\]

(ii) If \( f, g \in C^+ \) then \( \max (f, g) \in C^+ \)
and \( \min (f, g) \in C^+ \)

(iii) If \( f \in C^+ \) and \( \lambda > 0 \) then
\[
\lambda f \in C^+ \quad \text{and} \quad \int_{\Omega^n} (\lambda f)(x) \, dx = \lambda \int_{\Omega^n} f(x) \, dx
\]

and, as a direct consequence, if \( f, g \) are
integrable then so are:

\( f + g \), \( \lambda f \) with \( \lambda \in \mathbb{R} \), \( \max (f, g) \),
\( \min (f, g) \), \( |f| \), see \( H(x, 1) \).
Monotone Convergence Theorem (in $C^+$) If $f_n \geq 0$ and $f_n \to f$ a.e.
then $f \in C^+$ and 
$$\int_{\Omega} f_n(x) \, dx \to \int_{\Omega} f(x) \, dx$$

Proof. For each $k \in \mathbb{N}$, $f_k \in C^+$ implies
$$\text{If } \{g_k \}_m \text{ is a non-decreasing sequence of}
\text{piecewise cont. functions with compact support and}
\lim_{m \to \infty} g_k(x) = f_k(x) \text{ a.e. in } \Omega$$

If we do this for all $k$ we have the
"table" of functions

$$g_{1,1} \leq g_{1,2} \leq g_{1,3} \leq \ldots \leq g_{1,m} \leq \ldots \to f_1 \text{ a.e.}$$
$$g_{2,1} \leq g_{2,2} \leq g_{2,3} \leq \ldots \leq g_{2,m} \leq \ldots \to f_2 \text{ a.e.}$$
$$\vdots$$
$$g_{k,1} \leq g_{k,2} \leq g_{k,3} \leq \ldots \leq g_{k,m} \leq \ldots \to f_k \text{ a.e.}$$
The goal is to use the diagonal sequence \( \{g_k, k\} \) and show it converges a.e. to \( f \) and it is increasing, hence \( f \in C^+ \).

This is false in general (for example consider \( f = x \in [0, 1] \) and change \( g_{2, m} \) to the zero function for \( m \leq k \). This change does not affect the convergence of \( g_{2, m} \) to \( f_2 \)). However, if we make the columns of the "table" non-decreasing the issue is fixed.

Locate at the second line on the table.

Consider

\[
g_{2, m} (x) = \max \{ g_{1, m} (x), g_{2, m} (x) \}
\]

Now \( \tilde{g}_{2, n} (x) \leq \tilde{g}_{2, n+1} (x) \) and

\[
g_{2, n} (x) \leq \tilde{g}_{2, n} (x) \leq f_{2, n} (x) \text{ a.e.}
\]

hence \( \lim_{n \to \infty} \tilde{g}_{2, n} (x) = f_{2} (x) \) a.e.

Now replace the second line in the table.

Delete \( \{ \tilde{g}_{2, m} \} \) and redefine \( \tilde{g}_{2, m} \) by \( g_{2, m} \).

Continue with the third line

\[
g_{3, m} = \max \{ g_{2, m}, g_{3, m} \}
\]
until all the lines are replaced. The table is now nondecreasing. Table in the
direction. Consequently, \( \{ g_k \} \) is nondecreasing.
Consider now:
\[
E_k = \{ x \in \mathbb{R}^n : g_{k,m}(x) \rightarrow \infty \}
\]
\[
E_0 = \{ x \in \mathbb{R}^n : f_k(x) \rightarrow \infty \}
\]
and \( E = \bigcup_{j=1}^{\infty} E_j \). These means \( E = \emptyset \) and
\[
\forall x \notin E, \quad \exists_k \text{ s.t. } g_k(x) < \infty \quad \forall m \rightarrow \infty
g_k(x) < M_k \quad \forall k \text{ and } \lim_{k \rightarrow \infty} g_k(x) = f_0(x)
\]
Fix \( x \notin E \) and \( \varepsilon > 0 \). Then \( \exists_{k_0} \) \( \forall k \geq k_0 \):
\[
0 \leq f(x) - f_k(x) < \frac{\varepsilon}{2}
\]
and \( \exists_{m_0} \) such that
\[
0 \leq f_k(x) - g_{k,m}(x) < \frac{\varepsilon}{2}
\]
Choose \( k_0 = \max \{ k_0, m_0 \} \). Then
\[
0 \leq f(x) - g_{k,m}(x) \leq \varepsilon \quad \forall k \geq k_0
\]
In conclusion
\[ \lim_{n \to \infty} g_n(x) = f(x) \text{ on } U \setminus \{x\} \]

hence \( \{g_n\}_{n \geq 1} \) is a non-decreasing sequence of piecewise continuous completely supported functions. By def of Lebesgue integral w.r.t. \( u \) hence

\[ \lim_{n \to \infty} \int g_n, u (x) \, dx = \int f, u \, dx \]

But \( g_n, u \leq f, u \) a.e. so

\[ \int g_n, u \, dx \leq \int f, u \, dx \]

consequently (by passing to \( n \to \infty \))

\[ \int f \, dx \leq \lim_{n \to \infty} \int f_n \, dx \]

The opposite inequality follows from

\( f_x \leq f \) a.e. and lemma in Lecture 2.

Q.E.D.
**Corollary** \( Y = g_k \in C^+, \ g_k > 0 \) then the series \( \sum_{k=1}^{\infty} g_k \in C^+ \) and
\[
\int \sum_{k=1}^{\infty} g_k \, dx = \sum_{k=1}^{\infty} \int g_k \, dx
\]

**Proof.** Let \( f_m = \sum_{k=1}^{m} g_k \) apply the previous theorem to the increasing seq
\[
f_m \in C^+ \quad \text{QED}
\]

**Theorem** (Boppo - Levi) \( \text{If } \sum_{k=1}^{\infty} f_k > 0 \)

are integrable then:
\[
\int \left( \sum_{k=1}^{\infty} f_k \right) \, dx = \sum_{k=1}^{\infty} \int f_k \, dx
\]

i.e., if \( \sum_{k=1}^{\infty} \int f_k \, dx < \infty \) then \( \sum_{k=1}^{\infty} f_k \)

c integrable, otherwise \( Y = \int f - g \) with \( f \in C^+ \setminus C_t^0 \) and \( g \in C_t^0 \).
Proof. Note that if \( \varphi \) is integrable and \( \varepsilon > 0 \) one may choose \( f, g \in C^+_b \) such that

\[
\varphi = f - g \quad \text{and} \quad g > 0, \quad \int g \, dx < \varepsilon
\]

Moreover, if \( \varphi > 0 \) then \( f > 0 \). Indeed, let

\[
h_k > g \quad \text{a.e., the piecewise continuous}
\]

\[
\varphi = f - g = f - h_k - (g - g_k) = f_k - g_k
\]

Clearly \( g_k > 0, \int g_k \, dx < \varepsilon \) for \( k \) large enough and \( f_k, g_k \in C^+_b, f_k = f - h_k \geq f - g = \varphi > 0 \).

Returning to the theorem, for each \( \varphi_k \) (integrable) choose \( f_k, g_k \) such that

\[
\varphi_k = f_k - g_k, \quad f_k, g_k \geq 0, \quad 
\int g_k \, dx \leq \frac{1}{2^k}
\]

Then the series \( \sum_{k=1}^{\infty} g_k \) satisfies
The corollary above hence \( g = \sum_{k=1}^{\infty} g_k \) is in \( C^+ \) because

\[
\sum_{k=1}^{\infty} g_k \leq 1 \quad \forall n \in \mathbb{N}.
\]

Now \( \sum_{k=1}^{\infty} f_k \) also satisfies the corollary and is in \( C^+ \) if and only if \( \sum_{k=1}^{\infty} \int g_k \, dx < \infty \)

\[
\sum_{k=1}^{\infty} \int f_k \, dx = \sum_{k=1}^{\infty} \int g_k \, dx + \sum_{k=1}^{\infty} \int g_k \, dx
\]

\[
\leq \sum_{k=1}^{\infty} \int f_k \, dx + 1
\]

all in all

\[
y = \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} g_k = f - g, \quad f, g \in C^+
\]

on \( f \in C^+ \) when \( \sum_{k=1} f_k \) is unbounded and

\[
\int f \, dx = \int f \, dx - \int g \, dx = \sum_{k=1}^{\infty} \int f_k \, dx - \sum_{k=1}^{\infty} \int g_k \, dx
\]

\[
= \sum_{k=1}^{\infty} \int (f_k - g_k) \, dx = \sum_{k=1}^{\infty} \int f_k \, dx
\]
Monotone Convergence Theorem

If \( \psi_k \leq \psi_{k+1} \) for all \( k \in \mathbb{N} \), then

\[
\lim_{k \to \infty} \int_{\psi_k(x)} \psi_{k+1}(x) \, dx = \lim_{k \to \infty} \int_{\psi_k(x)} \psi_k(x) \, dx
\]

i.e., if \( \lim_{k \to \infty} \int_{\psi_k(x)} \psi_{k+1}(x) \, dx < \infty \) then

\( \psi(x) = \lim_{k \to \infty} \psi_k(x) \) is integrable and satisfies the above holds, otherwise \( \psi = f - g \), \( f \in C^+, g \in C^- \), and the identity holds because \( \lim_{k \to \infty} \int_{\psi} = \infty \).

Proof: Use \( \psi_1 = \psi_2 - \psi_1 \), ..., \( \psi_k = \psi_{k+1} - \psi_k \) in the preceding theorem.

Remark: A similar result holds for decreasing sequences \( \{ f_k \} \), \( f_k \in C^+ \), but if

\[
\lim_{k \to \infty} \int_{\psi_k(x)} \psi_{k+1}(x) \, dx = -\infty \text{ then}
\]

\( \psi = f - g \) with \( f \in C^+, g \in C^- \).
Remark. If \( \Phi_0 \) is integrable and
\[
L(\Phi_0) = \left\{ \Phi : \mathbb{R}^n \to \mathbb{R} \mid -\Phi_0 \leq \Phi \leq \Phi_0 \text{ a.e.} \right\}
\]
if \( \phi_k \uparrow \Phi \) on \( \mathbb{R} \times \mathbb{R} \) with \( F_0 \phi_k \leq \) \( L(\Phi_0) \),
then \( \Phi \in L(\Phi_0) \).

In particular, if \( \int \chi_k \phi_k \leq L(\Phi_0) \),
then
\[
\Phi = \sup \{ \phi_1, \phi_2, \ldots, \phi_k, \ldots \} \in L(\Phi_0) \quad \text{and}
\]
\[
\Psi = \inf \{ \phi_1, \phi_2, \ldots, \phi_k, \ldots \} \in L(\Phi_0)
\]
because
\[
\Phi_k = \max \{ \phi_1, \phi_2, \ldots, \phi_k \} \geq \Phi \quad \text{and} \quad \Phi_k \in L(\Phi_0)
\]
and
\[
\Psi_k = \min \{ \phi_1, \phi_2, \ldots, \phi_k \} \geq \Phi \quad \text{and} \quad \Psi_k \in L(\Phi_0)\]
**Dominated Convergence Theorem I**

If \( f_k \in L^+(\mathbb{R}) \) and \( f_k \to f \) a.e., then \( f \in L^+(\mathbb{R}) \) and

\[
\int f \, dx = \lim_{k \to \infty} \int f_k \, dx
\]

**Proof**

Let \( \underline{f}_k = \inf \{ f_k, f_{k+1}, \ldots \} \)

\[
\underline{f}_k = \sup \{ f_k, f_{k+1}, \ldots \}
\]

By the above remark \( \underline{f}_k, \overline{f}_k \in L^+(\mathbb{R}) \)

and\( \underline{f} \leq \overline{f} \) a.e., \( \overline{f}_k \to \overline{f} \) a.e \( (\text{the convergence holds at all points for which } f_k(x) \text{ converges to } f(x)). \)

By **Monotone Convergence Theorem**, we now have \( f \in L^+(\mathbb{R}) \) and

\[
\int \underline{f}_k \, dx \leq \int f_k \, dx \leq \int \overline{f}_k \, dx
\]

\[\overline{\int f \, dx} \]

hence \( \lim_{k \to \infty} \int f_k \, dx = \int f \, dx. \)
Corollary. If \( \phi_0 \) is integrable and \( \phi \) is measurable with

\[ -\phi_0(x) \leq \phi(x) \leq \phi_0(x) \quad \text{a.e. in } \Omega \]

then \( \phi \) is integrable and

\[ -\int \phi_0 \, dx \leq \int \phi \, dx \leq \int \phi_0 \, dx \]

Proof. Let \( f_k \rightarrow \phi \) a.e. with \( f_k \)

preceese continuous with compact support

let \( f_k = \max \left\{ -\phi_0, \min\left\{ f_k, \phi_0 \right\} \right\} \)

We have \( f_k \in L(\Omega) \) and \( f_k \rightarrow \phi \) a.e.

By the above dominated convergence 

therein \( \phi \in L(\Omega) \) and

\[ -\int \phi_0 \, dx \leq \int \phi \, dx \leq \int \phi_0 \, dx \]
**Dominated Convergence II:** If \( f_k \to f \) a.e. are measurable and \( f \) is integrable and \( |f_k(x)| \leq g(x) \) a.e. \( \forall k \in \mathbb{N} \), then \( f_k, f \) are integrable and
\[
\int f \, dx = \lim_{k \to \infty} \int f_k \, dx
\]

**Proof:** By Corollary above \( f_k \) are integrable. Apply now Dominated Convergence Theorem I. \( \text{Q.E.D.} \)
Fatou Lemma: If \( \psi_k > 0 \) are integrable then:
\[
\int \liminf_{k \to \infty} \psi_k \, dx \leq \liminf_{k \to \infty} \int \psi_k \, dx
\]

Remark: The definition of \( \liminf \) of a sequence \( \{a_n\} \in \mathbb{R} \) is:
\[
\liminf a_n = \inf \{\sup \{a_n, a_{n+1}, \ldots\} : n \to \infty\}
\]

\( b_n \) is nondecreasing so \( \lim b_n = \limsup_{n \to \infty} b_n \)

and by definition
\[
\liminf a_n = \lim b_n
\]

Proof of Fatou Lemma:

Let \( \psi_k(x) = \inf \{ \psi_k(x), \psi_{k+1}(x), \ldots \} \geq 0 \) then \( \psi_k \) is nonnegative and \( 0 \leq \psi_k \leq \psi_k \).

By Corollary above, \( \psi_k \) is integrable and \( 0 \leq \int \psi_k \leq \int \psi_k \).
Moreover, \( \Psi_k \not\leq \Psi \) and by definition

\[ \forall \Psi = \liminf_{k} \Psi_k. \]

By Monotone Convergence Theorem,

\[ \int \Psi \, dx = \lim_{k \to \infty} \int \Psi_k \, dx \]

But \( \int \Psi_k \, dx \leq \int \Psi_0 \, dx \quad \forall \, k \geq 1 \)

\[ \implies \int \Psi_k \, dx \leq \inf \{ \int \Psi_0 \, dx, \int \Psi_1 \, dx, \ldots \} \]

\[ \implies \int \Psi \, dx \leq \lim \inf_{k \to \infty} \int \Psi_k \, dx \]

by definition

Q.E.D.