1. (i) Show that the gravitational potential induced by a uniform distribution of masses on a sphere of radius $R$ is constant in the interior of the sphere while on the exterior of the sphere it is equivalent to the potential generated by a point of mass equal to the total mass and situated in the center of the sphere. Hint: the potential is a single layer potential with constant density on the sphere, show first that the potential must be radially symmetric hence constant on the sphere.

(ii) Show that the gravitational potential induced by a radially symmetric distribution of masses in a ball of radius $R$ is equivalent, in the exterior of the ball, to the potential induced by the entire mass situated at the center of the ball. Find a similar expression for the potential in the interior of the ball. Hint: use part (i) and spherical coordinates.

2. Consider the exterior Neumann problem for Poisson equation in the plane:

$$
\Delta u(x) = f(x), \quad x \in \mathbb{R}^2 \setminus \Omega,
$$

$$
\frac{\partial u}{\partial n}(x) = g(x), \quad x \in \partial \Omega
$$

$$
\sup_{|x|>R} |u(x)| < \infty
$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, with piecewise $C^1$ orientable boundary on which the normal derivative points inside the domain. $f \in L^\infty(\mathbb{R}^2)$ has compact support and $R > 0$ is sufficiently large so that the ball of radius $R$ includes both $\Omega$ and the support of $f$. Show that the classical solution of the problem is unique up to a constant, and the compatibility condition:

$$
\int_{\partial \Omega} g(y) dS_y = \int_{\mathbb{R}^2 \setminus \Omega} f(x) dx
$$

is necessary for the existence of solution. Hint: use a similar argument to the three dimensional case, see theorem 9.2 in Lecture 15, but note the differences due to the boundary condition at infinity.

3. Find the Green function for the Dirichlet problem in the half plane $\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}$ and in the disc $\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2 \}$. Write down the Poisson formula in each case. (Extra credit) How does the Poisson formula for the disc compare with Cauchy formula for holomorphic functions?

4. If $u$ is harmonic in the open set $\Omega \subseteq \mathbb{R}^n$ and the ball $B_r(x) = \{ y \in \mathbb{R}^n \mid |y - x| < r \}$ together with its closure is included in $\Omega$ then show that:

$$
\frac{\partial u}{\partial x_i}(x) = \frac{n}{\omega_n r^{n+1}} \int_{|y-x|=r} u(y) y_i dS_y \leq \frac{n}{r} \max_{|y-x|=r} |u(y)|.
$$

and deduce Liouville’s Theorem: bounded harmonic functions on $\mathbb{R}^n$ are constant.
Show that 
\[ |D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \int_{B_r(x)} |u(y)|dy \]
for each multiindex \( \alpha \), \(|\alpha| = k\), where \( C_0 \) is the inverse of the volume of the unit ball in \( \mathbb{R}^n \), while \( C_k = C_0 (2n+1)^k k^k \). Hint: use induction on \( k \) and the first part for \( r/2 \), then evaluate the maximum via the mean value theorem on balls of radius \( r/2 \) centered on the boundary of \( B_{r/2}(x) \). (Extra credit) Deduce that \( u \) is analytic on \( B_r(x) \), hence in \( \Omega \).

5. (Weak Maximum Principle for nonlinear Laplace eq) State and prove a weak maximum principle for \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfying the equation 
\[ \Delta u(x) - u^3 = 0, \quad x \in \Omega \subset \mathbb{R}^n, \]
where \( \Omega \) is bounded and open. Use it to show uniqueness and continuous dependence on data for the classical solution of the boundary value problem:
\[ \begin{align*}
\Delta u(x) - u^3 &= f(x), \quad x \in \Omega, \\
u(x) &= g(x), \quad x \in \partial \Omega
\end{align*} \]
Hint: use an argument similar to the one for the heat equation.

6. (Weak Maximum Principle in unbounded domains. Reflection Method) Consider the mixed (boundary and initial value) problem on the half-line:
\[ \begin{align*}
p_t u &= a^2 \partial_x^2 u + f(t,x), \quad \text{for } (t,x) \in \Omega = (0,T) \times (0,\infty), \\
u(t,0) &= g(t), \\
\lim_{x \to \infty} u(t,x) &= 0, \quad \text{uniformly in } t \in [0,T], \\
u(0,x) &= u_0(x).
\end{align*} \]
Do the following.
(i) State and prove a weak maximum principle for the solutions \( u \in C^{1,2}(\Omega) \cap C(\overline{\Omega}) \) for the problem above.
(ii) Use part (i) to show that the problem above has a unique classical solution, i.e. a unique solution in \( C^{1,2}(\Omega) \cap C(\overline{\Omega}) \), which depends continuously on the data \( f, g, u_0 \).
(iii) Let \( g(t) \equiv 0 \). Find a solution of the problem via the method of reflections, i.e. extend the data \( f, u_0 \) to odd functions on \( \mathbb{R} \) and use the theory for heat equation in the whole space to find a solution for the extended problem, see Lecture 17. Find sufficient conditions on \( u_0 \) and \( f \) such that this extended solution is classical, see Lecture 17, then use the symmetry to show that the boundary condition at \( x = 0 \) is always satisfied. Give an example of \( u_0 \neq 0, f \) for which the solution obtained from reflection method is classical and calculate it.

7. (Energy and Reflection Methods.) Consider the mixed (boundary and initial value) problem on a segment
\[ \begin{align*}
p_t u &= a^2 \partial_x^2 u + f(t,x), \quad \text{for } (t,x) \in \Omega = (0,\infty) \times (0,L), \\
u(t,0) &= g(t), \\
\partial_x u(t,L) &= h(t), \\
u(0,x) &= u_0(x).
\end{align*} \]
Prove the following.

(i) If \( f(t, x) = g(t) = h(t) \equiv 0 \), and \( u \in C^{1,2}(\Omega) \cap C^1(\overline{\Omega}) \) then the energy:
\[
E(t) = \int_0^L u^2(t, x) \, dx
\]
is decreasing in time.

(ii) Use part (i) to show that the problem above, even when \( f, g, h \), are arbitrary has a unique classical solution, i.e. a unique solution in \( C^{1,2}(\Omega) \cap C^1(\overline{\Omega}) \). (Note that the purpose of the extra smoothness \( u \in C^1(\Omega) \), instead of \( u \in C(\overline{\Omega}) \) from the lecture notes, is to make sense of the boundary condition at \( x = L \). Also note that because of the same boundary condition we could not have used the (weak) maximum principle to show uniqueness.)

(iii) Let \( g(t) \equiv 100 \), \( h(t) \equiv 2 \). Find a solution of the problem via the method of reflections. Be careful with the reflection method: first you want to have boundary conditions zero, then you want to extend the data \( f \), \( u_0 \) on the entire \( \mathbb{R} \) such that the extension enforces the boundary conditions at \( x = 0 \) and \( x = L \). For the first part think at subtracting the equilibrium solution \( u(t, x) = u(x) \) for the problem with no forcing and no initial data, i.e. it satisfies the Laplace equation and the boundary conditions. For the second part think what kind of symmetry forces a smooth function to be zero at a point, respectively to have zero derivative at a point. Find sufficient conditions on \( u_0 \) and \( f \) such that this extended solution is classical, then show that indeed the solution satisfies the boundary conditions at \( x = 0 \) and \( x = L \). Give an example of \( u_0 \), \( f \) for which the solution obtained from reflection method is classical and calculate it.

8. (Extra Credit) Show that any generalized (distributional) solution of:
\[
\partial_t u = a^2 \partial_x^2 u, \quad (t, x) \in \Omega \subset \mathbb{R}^{n+1}
\]
where \( \Omega \) is open and bounded must be generated by a \( C^\infty(\Omega) \) function. Hint: use an argument similar with that for harmonic functions.

9. Show that the distributions:
\[
\mathcal{E}_n = \frac{\theta(t)}{(4\pi it)^n} e^{-\frac{|x|^2}{4it}}
\]
are fundamental solutions of the Schrödinger operator: \( L(D) = \partial_t - i\Delta \). Then prove that for \( f(x) \) integrable, and square integrable on \( x \in \mathbb{R}^n \), we have for any \( \tau > 0 \):
\[
\sup_{y \in \mathbb{R}^n} |(\mathcal{E}_n * \delta(t) \cdot f(x))(\tau, y)| \leq \frac{1}{\tau^n} \int_{\mathbb{R}^n} |f(x)| \, dx,
\]
\[
\int_{\mathbb{R}^n} |(\mathcal{E}_n * \delta(t) \cdot f(x))(\tau, y)|^2 \, dy = \int_{\mathbb{R}^n} |f(x)|^2 \, dx.
\]

What are the implications of these results for the Schrödinger initial value problem:
\[
i \partial_t u(t, x) = -\Delta u(t, x), \quad u(0, x) = f(x)?
\]