1. In the previous homework you showed
\[ \frac{1}{\pi x} \sin \frac{x}{\epsilon} \xrightarrow{D'(\mathbb{R})} \delta, \quad \text{as } \epsilon \downarrow 0. \]
Now show that the convergence also holds in \( S'(\mathbb{R}) \). (This result was used in the argument that shows \( \frac{1}{(2\pi)^n} F^* = F^{-1} \) on \( S(\mathbb{R}^n) \), see Lecture 12. Alternatively one could use a density argument: first show \( F^* F(\phi) = (2\pi)^n \phi \) for all \( \phi \in D(\mathbb{R}^n) \) then use continuity of \( F \) and \( F^* \) together with the density of \( D(\mathbb{R}^n) \) in \( S(\mathbb{R}^n) \) to obtain \( F^* F(\phi) = (2\pi)^n \phi \) for all \( \phi \in S(\mathbb{R}^n) \). Finish the details of this alternative argument as an optional homework.)

2. Let \( \theta : \mathbb{R} \mapsto \mathbb{R} \) be the Heaviside function. Show that:
   (a) \( F[e^{-ik\cdot x}] = (2\pi)^n \delta(\xi - k) \), for any fixed \( k \in \mathbb{R}^n \),
   (b) \( F[\theta](\xi) = \pi \delta(\xi) - iP_1^\xi, \quad F[\theta(-x)] = \pi \delta - iP_1^\xi, \)
   (c) \( F[\text{sign}(x)] = 2iP_1^\xi, \quad F[P_1^x] = i\pi \text{sign}\xi, \)
   (d) \( \frac{d}{dx} P_1^\xi \phi = -P_1^\xi \phi, \quad F[P_1^x] = -\pi |\xi|, \quad F[|x|] = -2P_1^{\frac{1}{x^2}}, \) where \( (P_1^\xi, \phi) = \lim_{\epsilon \searrow 0} \int_{-\epsilon}^{-\xi} + \int_{\xi}^{\epsilon} \frac{\phi(x) - \phi(0)}{x^2} \, dx. \)

3. (Fourier Series) Show that:
   (i) If the coefficients \( \{a_k\}_{k \in \mathbb{Z}} \) satisfy \( |a_k| \leq C(1 + |k|)^m \) for some \( C > 0, \ m \in \mathbb{N} \), then the series:
      \[ \sum_{k=\infty}^{\infty} a_k \delta(\xi - k), \quad \sum_{k=\infty}^{\infty} a_k e^{-ikx}, \]
   are both convergent in \( S'(\mathbb{R}) \) and
   \[ F^{-1} \left[ \sum_{k=\infty}^{\infty} 2\pi a_k \delta(\xi - k) \right] = \sum_{k=\infty}^{\infty} a_k e^{-ikx}. \]
(ii) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally integrable and periodic with period $2\pi$ then we have $F(f)(\xi) = \sum_{k=-\infty}^{\infty} a_k \delta(\xi - k)$ where

$$a_k = \int_{-\pi}^{\pi} f(x) e^{ikx} \, dx.$$ 

Moreover, show that the Fourier series $\sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi} e^{-ikx}$ converges in $\mathcal{S}'(\mathbb{R})$ to the distribution generated by $f$.

4. This problem is optional: Show that for any $g \in \mathcal{E}'(\mathbb{R}^n)$ the map

$$x \mapsto (g(\xi), e^{i\xi \cdot x})$$

is in $C^\infty(\mathbb{R}^n)$. 