Math 553
Homework 5
Due April 17, 2014 before class

1. Let $\Omega \subseteq \mathbb{R}^n$ be open, $a \in C^\infty(\Omega)$, $f \in \mathcal{D}'(\Omega)$, and $\alpha, \beta \in \mathbb{N}^n$. Show that:

(i) $D^\alpha(D^\beta f) = D^{\alpha+\beta} f = D^\beta(D^\alpha f)$

(ii) $D^\alpha(a f) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta a D^{\alpha-\beta} f$

(iii) $\text{supp} D^\alpha f \subseteq \text{supp} f$.

2. Show that:

(i) $\frac{d}{dx} \ln |x| = P_1^1 x$.

(ii) the sequence of functions

$$f_n(x) = \sum_{k=1}^n (-1)^k \frac{4}{k^2} \cos(kx)$$

converges uniformly on $x \in \mathbb{R}$, to the $2\pi$-periodic function $f(x)$ where $f(x) = x^2 - \frac{\pi^2}{4}$, $x \in [-\pi, \pi)$, then find the distributional limit of the sequence $f''_n(x) = \sum_{k=1}^n (-1)^{k+1} \frac{4}{k} \cos(kx)$.

(iii) $u = c_1 + c_2 \theta(x) + c_3 \delta(x) - P_1^1 x$ solves $x^2 u' = 1$, where $\theta : \mathbb{R} \mapsto \mathbb{R}$ is the Heaviside function: $\theta(x) = 1$, for $x > 0$, and zero in rest.

(iv) $\frac{\partial^n [\theta(x_1)\theta(x_2)\cdots\theta(x_n)]}{\partial x_1 \partial x_2 \cdots \partial x_n} = \delta(x_1)\delta(x_2)\cdots\delta(x_n) = \delta(x)$.

3. Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{D}'(\mathbb{R}^m)$, and $\phi \in \mathcal{D}(\mathbb{R}^{n+m})$, $\psi \in \mathcal{D}(\mathbb{R}^n)$. Show that

(a) $x \mapsto (g(y), \phi(x,y))$ is in $\mathcal{D}(\mathbb{R}^n)$.

(b) $fg = gf$.

(c) $x \mapsto (f(y), \psi(x-y))$ is in $\mathcal{E}(\mathbb{R}^n)$, and $f * \psi = (f(y), \psi(x-y))$.

(d) $f * \omega_{\epsilon} = \omega_{\epsilon} * f$ are $C^\infty(\mathbb{R}^n)$ functions and $f * \omega_{\epsilon} \xrightarrow{\mathcal{D}'(\mathbb{R}^n)} f$ as $\epsilon \downarrow 0$ where $\omega_{\epsilon}(x) = \frac{1}{\epsilon^n} \omega_1\left(\frac{x}{\epsilon}\right)$ and $\omega_1$ is the Mexican hat function normalized to have integral 1.

(iv) $f_\alpha * f_\beta = f_\sqrt{\alpha^2 + \beta^2}$ where $f_\alpha = \frac{1}{\alpha \sqrt{2\pi}} e^{-\frac{x^2}{2\alpha^2}}$, $\alpha > 0$. 