1. Show that the problem
\[
\frac{\partial u}{\partial y} = \sin \left( \frac{\partial u}{\partial x} \right),
\]
\[u(x,0) = \frac{\pi x}{4},\]
has a unique analytical solution in a neighborhood of \((x,y) = (0,0)\) and calculate its power series. Is there another method to find solutions of this problem? If yes use it and compare the results.

2. Consider the initial value problem
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},
\]
\[u(0,x) = g(x).\]

   (a) Find the power series expansion of the solution around \((t,x) = (0,0)\) when \(g(x) = a_n x^n + \ldots + a_0\) is a polynomial. Where does the series converge?

   (b) If \(g(x) = \frac{1}{1-x^2}\) show that the problem has no real analytic solution in any neighborhood of \((0,0)\).

3. Use the \(\frac{\partial^2 u}{\partial \xi \partial \eta} = f(\xi, \eta, u, \partial_\xi u, \partial_\eta u)\) canonical form for hyperbolic equations and \(\frac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta, u, \partial_\xi u, \partial_\eta u)\) canonical form for parabolic equations to find the general solutions of the following equations:

   (i) \(\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\),

   (ii) \(\frac{\partial^2 u}{\partial x^2} - 2 \sin(x) \frac{\partial^2 u}{\partial x \partial y} - \cos^2(x) \frac{\partial^2 u}{\partial y^2} - \cos(x) \frac{\partial u}{\partial y} = 0\);

   (iii) \(y^2 \frac{\partial^2 u}{\partial x^2} - 2y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x} + 6y.\)
4. Show that the Monge-Ampère equation \( \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 = f(x) \), with \( f(x) > 0 \) is elliptic for any solution \( u \).

5. Consider the second order equation

\[
a(x,y) \frac{\partial^2 u}{\partial x^2} + 2b(x,y) \frac{\partial^2 u}{\partial x \partial y} + c(x,y) \frac{\partial^2 u}{\partial y^2} + f(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0.
\]

In class, see Lecture 8, we showed how to construct in general the families of characteristic curves: \( \Phi(x,y) = \text{constant}, \Psi(x,y) = \text{constant} \).

(i) In the hyperbolic case: \( b^2 - ac > 0 \) show that the change of variable \( \xi = \Phi(x,y), \eta = \Psi(x,y) \), is a local diffeomorphism.

(ii) In the parabolic case: \( b^2 - ac = 0 \) show that the change of variable \( \xi = \Phi(x,y), \eta = x \), is a local diffeomorphism and transforms the equation into: \( a \frac{\partial^2 u}{\partial \eta^2} + \tilde{f}(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) = 0 \).

(iii) In the elliptic case: \( b^2 - ac < 0 \) show that the change of variable \( \xi = \frac{\Phi(x,y) + \Phi(x,y)}{2}, \eta = \frac{\Phi(x,y) - \Phi(x,y)}{2i} \), is a local diffeomorphism.