§ 4.6. Power Series Method and
Caudy-Kovalevskaia Theorem

Consider the initial value (Caudy) problem:

\[
\frac{\partial^2 U}{\partial t^2} = f \left( t, x, U, \frac{\partial U}{\partial t}, \frac{\partial U}{\partial x}, \frac{\partial^2 U}{\partial t \partial x}, \frac{\partial^2 U}{\partial x^2} \right)
\]

\[ t \in \mathbb{R}, x \in U \subseteq \mathbb{R} \text{ open} \]

\[
U(0, x) = g_0(x) \quad x \in U
\]

\[
\frac{\partial U}{\partial t} (0, x) = g_1(x) \quad x \in U
\]

**Power series method** let \( x_0 \in U \) find the solution in a neighborhood \( V \) of \((0, x_0)\) in the form of a power series:

\[
U(t, x) = U(t_0, x_0) + U_{1,0}^1 (t-t_0) + U_{0,1}^1 (x-x_0) + U^2_{1,0} (t-t_0)^2 + U_{1,1}^1 (t-t_0)(x-x_0) + U_{0,2}^1 (x-x_0)^2 + \cdots
\]

\[
= \sum_{\delta, k=0} U_{\delta, k} (t-t_0)^\delta (x-x_0)^k
\]
Remark 1: If the series on the right converges on the open set \( V \), then its sum is called "real analytic function" on \( V \).

Real analytic functions are \( C^\infty \) and completely determined by its derivatives at a point.

In particular, for (2), we must have:

\[
(3) \quad u^i, j_k = \frac{\partial^{j+k} u}{(\partial t)^j (\partial x)^k} (0, x_0) = \frac{1}{j! \cdot k!}
\]

Formally, the coefficients \( u^i, j_k \) given by (3) can be computed from (1). Indeed,

\[
u^0, j_k = \frac{1}{k!} \frac{\partial^k u}{\partial x^k} (0, x_0) = \frac{1}{k!} g_0^{(k)} (x_0)
\]

\[
u^1, j_k = \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left( \frac{\partial u}{\partial t} \right) (0, x_0) = \frac{1}{k!} g_1^{(k)} (x_0)
\]

\[
u^2, j_k = \frac{1}{2!} \frac{\partial^2}{\partial t^2} u (0, x_0) = \frac{1}{2!} \int_0^t (0, x_0, u^{00}, u^{01}, u^{10}, u^{11}, 2! u^{02})
\]
\[ U^{2,1} = \frac{1}{2!} \frac{\partial}{\partial x} \left( \frac{\partial^2 U}{\partial t^2} \right) (0, x_0) \]

\[ = \frac{1}{2!} \frac{\partial}{\partial x} \left( f \right) (0, x_0) \]

\[ = \frac{1}{2!} \left[ \frac{\partial f}{\partial x} (0, x_0, u_0^{0,0}, u_0^{1,0}, u_0^{0,1}, u_0^{1,1}, 2! u_0^{0,2}) \right. \]

\[ + \left. \frac{\partial f}{\partial u} \left( \ldots \right) u_0^{0,1} + \ldots \right] \]

By induction we can compute \( U^{2,k} \) \( k \geq 0 \).

\[ U^{3,0} = \frac{1}{3!} \frac{\partial}{\partial t} \left( \frac{\partial^2 U}{\partial t^2} \right) (0, x_0) = \frac{1}{3!} \frac{\partial}{\partial t} \left( f \right) (0, x_0) \]

\[ = \frac{1}{3!} \left[ \frac{\partial f}{\partial t} (0, x_0, u_0^{0,0}, u_0^{1,0}, u_0^{0,1}, u_0^{1,1}, 2! u_0^{0,2}) + \right. \]

\[ + \left. \frac{\partial f}{\partial u} \left( \ldots \right) u_0^{1,0} + \ldots \right] \]

and by induction we can compute \( U^{3,k} \) \( k \geq 0 \) provided \( f \) is \( C^\infty \).
To show that the series (2) with coefficients calculated as above is convergent in a (small) neighborhood one uses the method of Majorants for power series, see course book. The method only requires:

1° $g_0, g_1$ are analytic in a neighborhood of $x_0$.

2° $f$ is analytic in a neighborhood of $(0, x_0, g_0(x_0), g_1(x_0), g_0'(x_0), g_1'(x_0), g_0''(x_0))$ and $f$ does not depend on higher than second order derivatives of $U$.

**Theorem** (Laplace-Convalescevlony) Consider

$$
\frac{d^k}{dt^k} = f(t, x_1, x_2, \ldots, x_{n-1}, U(t, x_1, x_2, \ldots, x_{n-1}), D^k U(t, x_1, x_2, \ldots, x_{n-1}))
$$

where $D^k U = \frac{\partial^k U}{\partial t^\alpha_0 \partial x_1^{\alpha_1} \ldots \partial x_{n-1}^{\alpha_{n-1}}}$, $\alpha=(\alpha_0, \ldots, \alpha_{n-1})$,

$$
|\alpha| = \alpha_0 + \alpha_1 + \ldots + \alpha_{n-1}, \quad |\alpha| \leq k-1
$$
With initial condition: \( x = (x_1, \ldots, x_{n-1}) \in U \subseteq \mathbb{R}^{n-1} \)

\[ U(0, x) = g_0(x) \]

\[ \frac{\partial U}{\partial t}(0, x) = g_1(x) \]

\[ \vdots \]

\[ \frac{\partial^{k-1}}{\partial t^{k-1}}(0, x) = g_{k-1}(x) \]

If \( g_0, \ldots, g_{k-1} \) are real analytic on \( U \) and \( f \) is real analytic in a neighborhood of \( (0, x_0, g_0(x_0), \ldots, g_{k-1}(x_0)) \) then there exists a neighborhood of \( (0, x_0) \) in which the equation has a unique real analytic solution satisfying the initial condition.
Remark 2. The theorem can be generalized to systems of partial differential equations:

\[
\frac{\partial^k_1 U_1}{\partial t^k_1} = f_1(t, x, U_1, ..., U_m, D^1 U_1, ..., D^m U_m)
\]

\[
\frac{\partial^k_2 U_2}{\partial t^k_2} = f_2(t, x, U_1, ..., U_m, D^1 U_1, ..., D^m U_m)
\]

\[\vdots\]

\[
\frac{\partial^k_m U_m}{\partial t^k_m} = f_m(t, x, U_1, ..., U_m, D^1 U_1, ..., D^m U_m)
\]

where \( k^0 = (k_1, k_2, ..., k_m) \), \( |k^0| \leq k_0 \), \( 0 \leq k_0 \leq k_0 - 1 \).

If \( f_1, ..., f_m \) are real analytic and given initial data

\[
\frac{\partial^i_0 U_1}{\partial t^i_0}(x_0) = g_{ij}(x) \quad 0 \leq i < k_0
\]

are analytic in a neighborhood of \( x_0 \) then the equations have a unique real analytic solution in a neighborhood of \((x_0, x)\) satisfying the initial conditions.
In fact the proof of Theorem reduces to an argument for a system that is first order in $t$:

$$(V_0, V_1, \ldots, V_{k-1}) = (V, \frac{\partial V}{\partial t}, \ldots, \frac{\partial^{k-1} V}{\partial t^{k-1}})$$

and such a reduction can be made also on the system of Remark 2.

**Examples**

1. (Hadamard) \[ \frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 V}{\partial x^2} = 0 \]

   \[ u(0, x) = 0 \]

   \[ \frac{\partial u}{\partial t} (0, x) = \frac{1}{k} \sin(kx) \]

   one can show using power series that

   \[ u(t, x) = \frac{e^{kt} - e^{-kt}}{2kt^2} \sin(kx) \]

   is the unique real analytic solution of the problem.

   Note that continuous dependence of data does not
\[ u(x) = \frac{1}{k^2} \sin kx \to 0 \text{ uniformly as } x \to \infty \]

but for any \( t > 0 \),

\[ \frac{e^{kt} - e^{-kt}}{2k^2} \sin kx \text{ does not converge to the zero function as } k \to \infty. \]

Example 2 (Korepanovskaya)

\[
\begin{cases}
\partial t u - \partial^2_x u = 0 \\
\partial x u = 0 \\
\partial x u (0, x) = \frac{1}{1 + x^2}
\end{cases}
\]

Does not satisfy the hypotheses of Theorem because the RHS of the eq depends on higher derivatives than LHS. Actually, the theorem does not even hold because the power series solution diverges for any \( t > 0 \).
The case of initial data on a hypersurface

Consider

\[
F\left(\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^2 u}{\partial x_j^2}, \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}, u, x_i, x_j\right) = 0
\]

with initial data along a curve \( \Gamma \):

\[
\left\{ \begin{array}{l}
x_i = x_i(s) \\
x_j = x_j(s)
\end{array} \right.
\]

\[
\left. u \right|_{\Gamma} = g_0(s)
\]

\[
\left. \frac{\partial u}{\partial s} \right|_{\Gamma} = g_1(s) \quad \text{where} \quad \vec{N} \quad \text{is the normal vector to} \quad \Gamma.
\]

Recall that a general curve, given by:

\[
\Gamma: \left\{ \begin{array}{l}
x_i = x_i(s) \quad \text{on interval in} \quad I \\
x_j = x_j(s)
\end{array} \right.
\]

\( \Gamma \) is \( C^1 \) if the functions \( x_i(s), x_j(s) \) are \( C^1 \) and

\[
\text{norme} \left[ \frac{dx_i}{ds}, \frac{dx_j}{ds} \right] = 1 \quad \text{for all} \quad s.
\]
Fix a point \((x_1^0, x_2^0)\) on the curve, corresponding to \(s = s_0\). The null condition implies that either
\[
\frac{dx_1(s_0)}{ds} \neq 0 \quad \text{or} \quad \frac{dx_2(s_0)}{ds} \neq 0
\]
If \(\frac{dx_2(s_0)}{ds} \neq 0\) (the other case is treated similarly)

Inverses function theorem applies in an interval containing \(s = s(s_2)\), hence in a neighborhood of \((x_1^0, x_2^0)\) we have:
\[
\begin{align*}
\Gamma & = x_1 = \tilde{x}_1(x_2) = x_1(s(s_2)) \\
\end{align*}
\]

Consider now the change of variable in \(\mathbb{R}^2\) in a neighborhood of \((x_1^0, x_2^0)\):
\[
\begin{align*}
\begin{cases}
  y & = x_1 - \tilde{x}_1(x_2) \\
  h & = x_2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial(y, h)}{\partial(x_1, x_2)} = \begin{bmatrix}
  1 & -\tilde{x}_1'(x_2) \\
  0 & 1
\end{bmatrix}
\end{align*}
\]
non-singular

In the new coordinates \(\Gamma\) \(y = 0\)
If \( \Gamma \) is \( C^2 \) then the change of coordinates (6) is \( C^2 \). Assuming \( \Gamma \) is \( C^2 \), we can change coordinates via (4), (5) to get

\[
\begin{cases}
F \left( \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u, x_1, x_2 \right) = 0 \\
u(x_1, x_2) = \tilde{f}_0(x_2) = \tilde{f}_1(x_2) \\
\frac{\partial u}{\partial x_1}(0, x_2) = \tilde{g}_1(x_2)
\end{cases}
\]

The point \((x_1, x_2)\) now corresponds to \((0, x_2)\).
As in the case of problem (1), all partial derivatives up to second order of \( u \) at \((0, x_2)\) except \( \frac{\partial^2 u}{\partial x_2^2} \) be calculated from initial data. Plugging in the equation:

\[
F \left( \frac{\partial^2 u}{\partial x_1^2}(0, x_2), \tilde{g}_1'(x_2), \tilde{g}_0(x_2), \tilde{g}_1(x_2), \tilde{g}_0'(x_2), \tilde{g}_1(x_2), 0, x_2 \right) = 0
\]

To start the power series method we need to find one solution \( \frac{\partial^2 u(0, x_2)}{\partial x_2^2} = \tilde{f}_0 \) of the above nonlinear eq in are unknown.
Def. We call the curve \( \gamma = 0 \) non-degenerate if for the eq. in (7) at the admisible:

\[
\frac{\partial^2 u}{\partial y^2} (0, \xi_0) = \tilde{\xi}_0
\]

If

\[
\frac{\partial \tilde{F}}{\partial \xi_1} (\tilde{\xi}_0, \tilde{g}_1 (\xi_0), \ldots) \neq 0
\]

Assuming \( \tilde{F} \) is \( C^{1} \) we can rewrite the eq in (7) via implicit function theorem:

\[
\frac{\partial^2 u}{\partial y^2} = \tilde{F} (\zeta, u, u, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial y^2})
\]

So (7) is equivalent to (6) and we can apply Coulby–Koebe–Nevanlinna Theorem provided \( \tilde{F}, \tilde{g}_0, \tilde{g}_1 \) are real analytic. But if \( \Gamma \) is real analytic then the change of coordinates (6) is real analytic and

\[
\tilde{F}, \tilde{g}_0, \tilde{g}_1 \text{ real analytic } \Rightarrow \tilde{F}, g_0, g_1 \text{ real analytic}
\]
Remark 3. The non-characteristic condition (8) can be checked directly on (4), (5). Denoting

\[ F = F (a_{11}, a_{12}, a_{22}, \nu, \eta, \zeta, x, y) \]

and the normal vector to \( \Gamma \):

\[ \mathbf{V} = (V_1, V_2) \]

\[ (8) = \frac{\partial F}{\partial a_{11}} V_1^2 + \frac{\partial F}{\partial a_{12}} V_1 V_2 + \frac{\partial F}{\partial a_{22}} V_2^2 \neq 0 \]

where the components \( V_1, V_2 \) are calculated at \((x_0, x_0)\) and the partial derivatives \( \frac{\partial F}{\partial a_{ij}} \) are calculated at an admissible \((a_{11}^0, a_{12}^0, a_{22}^0, \nu_0, \eta_0, \zeta_0, x_0, x_0)\).

Remark 4. In example 2, the surface \( t = 0 \) does not satisfy (3).

Remark 5. The entire discussion can be generalized for a partial differential equation (or a system) with an unknown function depending on \( u \)-variables with an initial condition along an \( u \)-1 hypersurface. See course notes for a rigorous complete description of the general case.