§ 3.1 Complete integrals, general solutions for first order PDE’s.

Consider first the case of quasilinear eq.

\[ a(x, y, u) \frac{\partial u}{\partial x} + b(x, y, u) \frac{\partial u}{\partial y} = c(x, y, u) \]  

Recall that the graph of any solution \( (x, y, u(x, y)) \) \( (x, y) \in \mathbb{R}^2, u \in \mathbb{R} \) is an integral surface for the vector field

\[ \nabla = (a(x, y, z), b(x, y, z), c(x, y, z)) \]

**Def.** If two \( C^1 \) functions \( \Phi, \Psi : V \to \mathbb{R}, V \subset \mathbb{R}^3 \) open have been found such that for all \( c_1, c_2 \in \mathbb{R} \)

\[ \Phi(x, y, z) = c_1 \]

\[ \Psi(x, y, z) = c_2 \]

are integral surfaces for \( \nabla \), and

\[ \nabla \Phi \neq \nabla \Psi \] at any point \( (x, y, z) \in V \)

then \( \Phi, \Psi \) is called a system of complete integrals.
Remark 1: The equation

\[ C_2 \Delta \Phi(x, y, z) - C_1 \Psi(x, y, z) = 0, \quad (C_1, C_2) \neq (0, 0) \]

gives a family of integral surfaces for \( \nabla^2 \)
which depends on two arbitrary parameters.

**Theorem 1** Let \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) be \( C^1 \) such that \( \nabla^2 F \neq 0 \) everywhere. Then the eq:

\[ F(x, y, z) = F(\Phi(x, y, z), \Psi(x, y, z)) = 0 \]

gives an integral surface for \( \nabla^2 \).

**Proof** \( \nabla F = \frac{\partial F}{\partial \Phi} \nabla \Phi + \frac{\partial F}{\partial \Psi} \nabla \Psi = 0 \) if and only if

\[ \frac{\partial F}{\partial \Phi} = \frac{\partial F}{\partial \Psi} = 0 \text{ impossible } \Rightarrow F = 0 \text{ gives a } C^1 \text{ surface} \]

\[ \nabla^2 \nabla F = \frac{\partial}{\partial \Phi} \left( \nabla^2 \nabla \Phi \right) + \frac{\partial F}{\partial \Phi} \left( \nabla^2 \nabla \Psi \right) = 0 + 0 = 0 \]

\[ \Rightarrow F = 0 \text{ is an integral surface for } \nabla^2 \text{. Q.E.D.} \]
Def 2: A family of integral surfaces for \( \nabla \Phi \) depends on an arbitrary function is called a general solution of (1)

Finding families of integral surfaces

Recall that \( \Phi \) is an integral surface for (1) iff

\[ \nabla \Phi \cdot (\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)) = 0 \]

So we require \((\alpha_1(x, y, z), \alpha_2(x, y, z), \alpha_3(x, y, z))\) such that:

\[ \alpha_1 \, dx + \alpha_2 \, dy + \alpha_3 \, dz \]

is an exact differential (i.e. \( \exists \Phi : \nabla \Phi = (\alpha_1, \alpha_2, \alpha_3) \))

and \( \alpha_1 \alpha_2 + \alpha_2 \beta + \alpha_3 \gamma = 0 \).

The algebra can be tracked using the characteristic system:

\[ \frac{dx}{\alpha_1} = \frac{dy}{\alpha_2} = \frac{dz}{\alpha_3} \Rightarrow \alpha_1 \, dx + \alpha_2 \, dy + \alpha_3 \, dz = 0 \]

\( \alpha(x_1, y_1, z_1), \beta(x_1, y_1, z_1), \gamma(x_1, y_1, z_1) \) iff \( \alpha_1 \alpha_2 + \alpha_2 \beta + \alpha_3 \gamma = 0 \)

\[ \Rightarrow \Phi(x, y, z) = C_1 \text{ is a family of integral surfaces} \]
Also recall that in general
\[ L(x_1, \ldots, x_n) \, dx_1 + L_2(x_1, \ldots, x_n) \, dx_2 + \cdots + L_n(x_1, \ldots, x_n) \, dx_n \]
is an exact differential iff

\[ \frac{\partial L_i}{\partial x_j} = \frac{\partial L_j}{\partial x_i} \text{ for all } i \neq j \tag{2} \]

and if (2) hold in a simply connected domain \( V \subseteq \mathbb{R}^n \) then \( \Phi : V \to \mathbb{R} \) satisfying

\[ \frac{\partial \Phi}{\partial x_i}(x) = L_i(x) \text{ for all } x \in V \text{ and } i = 1, \ldots, n \]
can be found via integration along any path in \( V \):

\[ \Phi(x) = \int_Y L_1 \, dx_1 + \cdots + L_n \, dx_n \]

where \( Y \in V \) is any path connecting a fixed \( x_0 \in V \) to \( x \in V \).

*Note.* Once a family of integral surfaces has been produced, \( \Phi(x_1, x_2, x_3) = C_1 \),
eliminate one variable from the characteristic system and find the second family of integral surfaces by solving the reduced system.

For example:

\[ \Phi(x, y, z) = c_1 \Rightarrow z = \Phi(x, y, c_1) \]

\[ \frac{dx}{a(x, y, \Phi(x, y, c_1))} = \frac{dy}{b(x, y, \Phi(x, y, c_1))} \]

\[ \Rightarrow \Phi(x, y, c_1) = c_2 \]

\[ \Rightarrow \Phi(x, y, z) = \Phi(x, y, \Phi(x, y, z)) \]

is a family of integral surfaces for (1) which completes the system of integral surfaces.

**Example 1**

\[ u \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \]


We Characteristic system:

\[ \frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x} \]

Choose \( l_1 = x, l_2 = 0, l_3 = -z \).
Hence \[ x \, dx - 2 \, dt = 0 \Rightarrow x^2 - 2^2 = C_1, \] one integral surface \( \Phi(x, y, z) = x^2 - z^2, \) \( z = \sqrt{x^2 - C_1} \)

\[ \frac{dx}{\sqrt{x^2 - C_1}} = \frac{dy}{y} \Rightarrow \ln|x + \sqrt{x^2 - C_1}| + C_2 = \ln|y| \]

or \( C_3 = \left| \frac{y}{x + \sqrt{x^2 - C_1}} \right| \) hence \( y = C_3 \) is a second family

of integral surfaces \( \Psi(x, y, z) = \frac{y}{x + z} \)

The general solution of the equation is:

\[ F\left(x^2 - y^2, \frac{y}{x + y} \right) = 0 \quad F \in C^1 \]

The one solution \( u(x, 1) = 2x \), see Example 1

Lecture 2 requires:

\[ F\left(x^2 - 4x^2, \frac{1}{x + 2x} \right) = 0 \quad \text{for all } x \in \mathbb{R} \]

The natural choice is then:
\[ F(\phi, \psi) = 3\phi + \psi^{-2} \]

\[ = \frac{3(x^2 + u^2) - (x + u)^2}{y^2} = 0 \]

\[ = (x + u)(3x - 3u + \frac{x + u}{y^2}) = 0 \]

\[ x + u = 0 \implies u = -x \text{ does not satisfy } \phi(x, 1) = 2x. \]

\[ 3x - 3u + \frac{x + u}{y^2} = 0 \implies u = x \frac{\frac{3y^2 + 1}{3y^2 - 1}} \text{ compare to lecture 2.} \]

Example 2. \[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \]

\[ \frac{dt}{1} = \frac{dx}{z} = \frac{dz}{0} \]

\[ \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1 \implies d\bar{z} = 0 \implies \bar{z} = C_1, \phi(t, x, z) = 2 \]

\[ \frac{dt}{1} = \frac{dx}{C_1} \implies x - C_1t = C_2 \]

\[ \implies x - 2t = C_2, \psi(t, x, z) = x - 2t \]
General case: \( F(U, x-ut) = 0 \) in simplified form:

\( U = f(x-ut) \)

The solution with initial data \( U = u(x) \) at \( t = 0 \) is obtained by making \( f = u \).

Generalization to \( n \)-dim nonlinear case:

\[ a_1(x_1, \ldots, x_n, U) \frac{\partial U}{\partial x_1} + \cdots + a_n(x_1, \ldots, x_n, U) \frac{\partial U}{\partial x_n} = a_{n+1}(x_1, \ldots, x_n, U) \]

Find \( n \) integral hypersurfaces \( \Phi_1, \Phi_2, \ldots, \Phi_n \), independent, i.e.

\[ \bigcap \{ \nabla \Phi_1, \ldots, \nabla \Phi_n \} = \emptyset \text{ everywhere} \]

by finding \( a_1, a_2, \ldots, a_{n+1} \) such that

\[ a_1(x_1, \ldots, x_n, \Phi_1) \, dx_1 + \cdots + a_{n+1}(x_1, \ldots, x_n, \Phi_n) \, dx_n \]

is an exact differential (of say \( \Phi_1 \)) and

\[ \sum_{i=1}^n \frac{\partial a_i}{\partial x_i} = 0. \]
Then \( \Phi_1, \ldots, \Phi_n \) form a complete system of integrals for (3). The general solution is

\[
F(\Phi_1, \ldots, \Phi_n) = 0
\]

where \( F \in C^1 \), \( \nabla F \neq 0 \) everywhere is arbitrary.

Remark 1 bis

\[
\sum_{j=1}^{n-1} (C_{f_j} \Phi_j - C_j \Phi_{j+1}) = 0
\]

gives an integral hypersurface that depends on \( n \) arbitrary constants from which, via implicit function theorem one can extract a solution depending on \( n \) arbitrary constants

\[
U = U(x_1, \ldots, x_n, C_1, \ldots, C_m)
\]
Generalization to fully nonlinear case

Unfortunately, the analog of Theorem 1 is no longer valid for

\( F(Du, u, x) = 0 \quad x \in \mathbb{R} \subset \mathbb{R}^n \) open

but the idea behind Remark 1 works:

Def 3. A complete integral for (3) is a \( C^2 \) solution \( u(x, a) \) of (3) depending on \( n \) arbitrary parameters \( a = (a_1, a_2, \ldots, a_n) \) such that

\[
\begin{bmatrix}
D^2 u \\
D^2 u
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 u}{\partial a_1 \partial x} & \frac{\partial^2 u}{\partial a_2 \partial x} & \cdots & \frac{\partial^2 u}{\partial a_n \partial x} \\
\frac{\partial^2 u}{\partial a_1 \partial x} & \frac{\partial^2 u}{\partial a_2 \partial x} & \cdots & \frac{\partial^2 u}{\partial a_n \partial x} \\
\frac{\partial^2 u}{\partial a_1 \partial x} & \frac{\partial^2 u}{\partial a_2 \partial x} & \cdots & \frac{\partial^2 u}{\partial a_n \partial x}
\end{bmatrix}
\]

has rank \( n \) everywhere.

Example 3. \( x \cdot Du + f(Du) = 0 \)

\[ u(x, a) = a \cdot x + f(a) \]
4) $|bu| = 1$

$$U(x, a, b) = a \cdot x + b \quad a \in DB(0, 1) \subset \mathbb{R}, \quad b \in \mathbb{R}$$

5) $U_t + H(u) = 0$

$$U(t, x, a, b) = a \cdot x - t + (a) + b$$

**Remark 2.** From a complete integral we can extract a new solution called the "envelope" by eliminating the constants in the equations.

$$\partial_a U(x, a) = 0 \Rightarrow a = a(x)$$,

$$U(x) = U(x, a(x))$$ is the envelope.

**Examples.**

3) $x_1 \frac{\partial U}{\partial x_1} + x_2 \frac{\partial U}{\partial x_2} + \left( \frac{\partial U}{\partial x_1} \right)^2 + \left( \frac{\partial U}{\partial x_2} \right)^2 = 0$

$$U(x_1, x_2, a_1, a_2) = a_1 x_1 + a_2 x_2 + a_1^2 + a_2^2$$

$$0 = \frac{\partial U}{\partial a_1} = x_1 + 2a_1$$,

$$0 = \frac{\partial U}{\partial a_2} = x_2 + 2a_2$$

$$U(x_1, x_2) = -\frac{x_1^2}{2} - \frac{x_2^2}{2} + \frac{x_1^2}{4} + \frac{x_2^2}{4} = -\frac{1}{4} \left( x_1^2 + x_2^2 \right)$$
Def 4. A family of solutions depending on an arbitrary function \( f: \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) is called the general solution of (3).

Remark 3. From a complete integral we can get the general solution by considering the envelope \( V(x, \alpha) \) of

\[
V(x, \alpha) = \varphi(x, \alpha, \ldots, \alpha_{n-1}, h(\alpha, \alpha_2, \ldots, \alpha_{n-1}))
\]

Example 5) \( V_t + \left( \frac{\partial V}{\partial \alpha} \right)^2 = 0 \)

\[
\varphi(t, x, \alpha, \beta) = \alpha x - t \alpha^2 + \beta
\]

\( \beta = \varphi(t, \alpha) = \alpha \) (particular choice for \( \beta \))

\[
\varphi(t, x, \alpha) = \alpha x - t \alpha^2 + \varphi(t, \alpha)
\]

\[
0 = \frac{\partial \varphi}{\partial \alpha} = x - 2 \alpha t + 1 \Rightarrow \alpha = \frac{x - 1}{2t}
\]

\[
\Rightarrow V(t, x, \alpha(t)) = \frac{x(x+1)}{2t} - \frac{(x+1)^2}{4t} + \frac{x-1}{2t}
\]

\[
= \frac{(x+1)^2}{4t}
\]