Summary

- Method of Energy for wave equation: uniqueness, continuous dependence on data, finite propagation speed.

- Method of Reflections and application to wave equation on half-line and on a segment.

1. Method of Energy:

Let \( \mathcal{U} \subset \mathbb{R}^n \) open, bounded and \( \partial \mathcal{U} \) piecewise \( C^1 \).
Assume \( U \in C^2((0, \infty) \times \mathcal{U}) \cap C^1([0, \infty) \times \overline{\mathcal{U}}) \) is a solution of:

\[
\begin{align*}
\begin{cases}
U_{tt} - c^2 \Delta U &= f(t, x) \quad t > 0, \ x \in \mathcal{U} \\
U(0, x) &= U_0(x), \quad U_t(0, x) = U_1(x)
\end{cases}
\end{align*}
\]

We can always find the bounded open sets \( \mathcal{U} \) with piecewise \( C^1 \) boundary such that \( \mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U} \), \( 0 < \varepsilon_2 < \varepsilon_1 \), and \( U \mathcal{U}_2 = \mathcal{U} \). (Why?)
Multiply the equation (1) by $u_0$ and integrate $t \in [0, T], x \in \Omega_e$:

$$\int_{\Omega_e} \int_0^T \left( \frac{\partial u}{\partial t} \right)^2 \, dt \, dx - a^2 \int_{\Omega_e} \frac{\partial u}{\partial t} \partial u \, dx \, dt$$

$$= \int_{\Omega_e} \int_0^T u_0 \eta(x, t) \, dx \, dt$$

We have:

$$\int_{\Omega_e} \int_0^T \left( \frac{\partial u}{\partial t} \right)^2 \, dt \, dx = \int_{\Omega_e} \left( \frac{\partial u}{\partial t} \right)^2 (T, x) \, dx - \int_{\Omega_e} \left( \frac{\partial u}{\partial t} \right)^2 (0, x) \, dx$$

$$\Rightarrow \quad \int_{\Omega_e} \left( \frac{\partial u}{\partial t} \right)^2 (T, x) \, dx = \int_{\Omega_e} \left( \frac{\partial u}{\partial t} \right)^2 (0, x) \, dx$$

$$\int_{\Omega_e} \int_0^T \partial u \partial u \, dx \, dt = - \int_{\Omega_e} \int_0^T \frac{\partial}{\partial t} (\nabla u) \cdot \nabla u \, dx \, dt$$

$$+ \int_{\Omega_e} \int_0^T \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} \, dx \, dt$$

$$\Rightarrow - \int_{\Omega_e} \int_0^T \left( \frac{\partial u}{\partial t} \right)^2 (T, x) \, dx + \int_{\Omega_e} \int_0^T \left( \frac{\partial u}{\partial t} \right)^2 (0, x) \, dx$$

$$\Rightarrow - \frac{1}{2} \int_{\Omega_e} \left( \nabla u \right)^2 (T, x) \, dx + \frac{1}{2} \int_{\Omega_e} \left( \nabla u \right)^2 (0, x) \, dx + \int_{\Omega_e} \int_0^T \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} \, dx \, dt$$
If

\[ u_t (t, x) = 0 \quad t > 0, \quad x \in \partial \Omega \]

or

\[ \frac{\partial u}{\partial n} (t, x) = 0 \quad t > 0, \quad x \in \partial \Omega \]

then the boundary term disappears. If for some \( a(x), \beta(x) \geq 0 \):

\[ a(x) u_t (t, x) + \beta(x) \frac{\partial u}{\partial n} (t, x) = 0 \quad t > 0, \quad x \in \partial \Omega \]

then:

\[ \int_0^T \int_{\partial \Omega} \frac{\partial u}{\partial n} \frac{\partial u}{\partial n} \, ds \, dt = \int_0^T \int_{\partial \Omega} - \frac{1}{2} \frac{1}{\beta} \frac{\partial}{\partial t} \left[ u^2 \right] \, ds \, dt \]

\[ = - \frac{1}{2} \int_{\partial \Omega} \frac{\partial}{\partial t} \left( u^2(t, x) \right) \, ds_x + \frac{1}{2} \int_{\partial \Omega} \frac{1}{\beta} \frac{\partial u^2}{\partial t} (x) \, ds_x \]

where \( \partial \Omega \) is the part of the boundary where \( \beta > 0 \).

All in all we get:

**Theorem 1 (conservation of energy)**: If \( \Omega \subseteq \mathbb{R}^n \)

is bounded, open, with \( \partial \Omega \) piecewise \( C^1 \) and \( u \) is a classical solution of (1) which satisfies:

(i) \( u(0, x) = 0 \quad t > 0, \quad x \in \partial \Omega \)

or (ii) \( \frac{\partial u}{\partial n} (t, x) = 0 \quad t > 0, \quad x \in \partial \Omega \)

(ii) \( \alpha(x) u_t (t, x) + \beta(x) \frac{\partial u}{\partial n} (t, x) = 0 \quad t > 0, \quad x \in \partial \Omega \)
Then:

\[ y(t) = y(0) + \int_0^T \int_{\Omega} \nabla \cdot \mathbf{f}(t, x) \, dx \, dt. \]

Where \( u \) in the case (i) and (ii):

\[ y(t) = \frac{1}{2} \int_{\Omega} \left( \frac{\partial u}{\partial t} \right)^2 \, dx + \frac{\kappa^2}{2} \int_{\Omega} |\nabla u|^2 \, dx. \]

Here for (iii):

\[ y(t) = \frac{1}{2} \int_{\Omega} \left( \frac{\partial u}{\partial t} \right)^2 \, dx + \frac{\kappa^2}{2} \int_{\Omega} |\nabla u|^2 \, dx \]

\[ + \frac{\kappa^2}{2} \int_{\Sigma} \frac{1}{\sqrt{\nu}} u^2 (t, x) \, d\Sigma. \]

Remark 1: The theorem can be obtained for more general hyperbolic operators, see textbooks §29.4 (pages 388–390), and under less restrictive conditions on \( u \) and boundary of \( \Omega \), see [10], [24].
Corollary 1 (Uniqueness of BVP) For \( \Omega \subset \mathbb{R}^n \) open, bounded with \( \partial \Omega \) piecewise \( C^1 \), problem (1) has a unique classical solution satisfying

\[
\begin{align*}
(i) \quad U(t, x) &= g(t, x) \quad t > 0, x \in \Omega \\
(ii) \quad \frac{\partial U}{\partial n} (t, x) &= g(t, x) \quad t > 0, x \in \partial \Omega \\
(iii) \quad d(x) U(t, x) + b(x) \frac{\partial U}{\partial n} (t, x) &= g(t, x) \quad t > 0, x \in \partial \Omega \\
\end{align*}
\]

Proof. Let \( \tilde{U}_1, \tilde{U}_2 \) be two such solutions then

\[
U = \tilde{U}_1 - \tilde{U}_2
\]

satisfies (i) with \( \int = 0, U_0 = 0, U_1 = 0 \) and zero boundary conditions. From the previous theorem we get

\[
\tilde{j}(t) = 0 \quad t \geq 0.
\]

\[
\Rightarrow \quad \frac{\partial U}{\partial t} = 0 \quad \text{and} \quad \nabla U = 0 \quad \text{on} \quad [0, T] \times \overline{\Omega}
\]

\[
\Rightarrow \quad U = \text{constant} \quad \text{on} \quad [0, T] \times \overline{\Omega} \Rightarrow U \equiv 0 \quad \text{since} \quad U(0, x) = 0 \quad \text{Q.E.D.}
\]
Corollary 2. Under the hypotheses of Corollary 1, the solution of the problem depends continuously on \( f, u_0 \) and \( u_1 \).

Proof. See textbook §29.2 (pages 391-394).

Corollary 3. (Domain of dependence.) Assume \( u \) is a \( C^2 \) solution of (1) on \( \Omega = \mathbb{R}^n \times (0, \infty) \times \mathbb{R}^n \). If \( f(t, x) \equiv 0 \) in the cone \( \overline{\Gamma(t_0, x_0)} = \{ (t, x) \mid |x - x_0| < a(t - t_0) \} \) and \( u_0 = u_1 \equiv 0 \) on \( B(x_0, a t_0) \) then

\[
    u(t, x) \equiv 0 \quad \text{on} \quad (t, x) \in \overline{\Gamma(t_0, x_0)}
\]

Proof. Define the local energy:

\[
    J(t) = \frac{1}{2} \int_{B(x_0, a(t_0-t))} \left( \frac{\partial u}{\partial t} \right)^2 + a^2 |\nabla u|^2 \, dx
\]

\[
    t \quad \big| \quad (t_0, x_0)
\]

\[
    B(x_0, a(t_0-t))
\]

\[
    B(x_0, a t_0)
\]
\[
\frac{d}{dt} y(t) = \int_{B(x_0, a(t_0-t))} v_0 u_0 + a^2 \nabla u \cdot \nabla v_0 \, dx - a \int_{\partial B(x_0, a(t_0-t))} v_0 \frac{\partial u_0}{\partial n} \, dS_x
\]

\[
= \int_{B(x_0, a(t_0-t))} v_0 (u_0 - a^2 u) \, dx + a^2 \int_{\partial B(x_0, a(t_0-t))} v_0 \frac{\partial u}{\partial n} \, dS_x - a \int_{\partial B(x_0, a(t_0-t))} v_0^2 + a^2 |\nabla v_0|^2 \, dS_x
\]

\[
= \frac{a}{2} \int_{\partial B(x_0, a(t_0-t))} 2a u_0 \frac{\partial u_0}{\partial n} - u_0^2 - a^2 |\nabla u|^2 \, dS_x
\]

But from \( \left| \frac{\partial u}{\partial n} \right| \leq |\nabla u| \) we get

\[
(2a u_0 \frac{\partial u_0}{\partial n} ) \leq u_0^2 + a^2 |\nabla u|^2
\]

\[
\Rightarrow \quad \frac{d}{dt} y(t) \leq 0 \Rightarrow y(t) \leq y(0) = 0 \quad \forall 0 \leq t \leq t_0
\]

\[
\Rightarrow \quad u_0, \nabla u = 0 \Rightarrow u = 0 \text{ on } \overline{B(x_0, a(t_0-t))}.
\]
Remark 2 Using Corollary 3 we can show uniqueness of bounded solutions of (1) for \( N = 1 \). Indeed, the difference of two Dirichlet solutions satisfies (1) with \( f = 0 \), \( u_0 = u_1 = 0 \) everywhere.

Method of Reflections, vorszeg on semi-infinite line.

Consider the Dirichlet problem:

\[
\begin{cases}
   U_{tt} - \alpha^2 U_{xx} = f(t, x) & t > 0, \ x > 0 \\
   U(0, x) = U_0(x); \ U_t(0, x) = U_1(x) & x > 0 \\
   U(t, 0) = 0 & t > 0
\end{cases}
\]

and the reflected problem (odd reflection with respect to \( x = 0 \)):

\[
\begin{cases}
   \tilde{U}_{tt} - \alpha^2 \tilde{U}_{xx} = \tilde{f}(t, x) & t > 0, \ x \in \mathbb{R} \\
   \tilde{U}(0, x) = \tilde{U}_0(x); \ \tilde{U}_t(0, x) = \tilde{U}_1(x) & x \in \mathbb{R}
\end{cases}
\]
where
\[ \overline{l}(t,x) = \begin{cases} l(t,x) & \text{if } x > 0 \\ -l(t,-x) & \text{if } x < 0 \end{cases} \]

(3)
\[ \overline{u}_0(x) = \begin{cases} u_0(x) & \text{if } x > 0 \\ -u_0(-x) & \text{if } x < 0 \end{cases} \]

\[ \overline{u}_1(x) = \begin{cases} u_1(x) & \text{if } x > 0 \\ -u_1(-x) & \text{if } x < 0 \end{cases} \]

**Theorem 2.** Problem (2) has a unique classical solution, \( \overline{u}_0 \in C^2(\mathbb{R}) \), \( \overline{u}_1 \in C^1(\mathbb{R}) \) and \( \overline{l} \in C^1(0,\infty) \times \mathbb{R} \) then the solution is given by the limit

(2') \[ U(b,t,x) = \overline{u}(b,t,x) \quad \text{for } t \geq 0, x > 0 \]

(4)
\[ \overline{u}(b,t,x) = \frac{\overline{u}_0(x-at) + \overline{u}_0(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \overline{u}_1(y) \, dy \]

\[ + \frac{1}{2a} \int_{-x}^{x-at} \overline{l}(t, y) \, dy \, dt \quad \text{for } t > 0, x \in \mathbb{R} \]
Proof. Uniqueness follows from the method of energy, see Remark 2. Existence follows from Conclusion (i) in Lecture 16 together with

$$\nabla (t, 0) = \frac{\nabla_0 (-\alpha t) + \nabla_0 (\alpha t)}{2} + \frac{1}{2a} \int_{-\alpha(t, \alpha)}^{\alpha(t, \alpha)} \xi_t (y) \, dy \tag{10}
$$

because $\nabla_0$ is odd, $\alpha$ is odd,

$$+ \frac{1}{2a} \int_{-\alpha(t, \alpha)}^{\alpha(t, \alpha)} \xi_t (y) \, dy \, dx \tag{11}
$$

because $f$ is odd in $y$.

$$= 0. \tag{12}
$$

Remark 3 Formula (4) gives a unique selection of (2) if $f$, $\nabla_0$ $\xi_t$ defined by (3) are locally integrable and $\nabla_0$ is absolutely continuous (i.e. $\nabla_0 \in L^1 (\mathbb{R})$), or, equivalently $f$, $\nabla_0$ are locally integrable and $\nabla_0$ is absolutely continuous with limit $\nabla_0 (x) = 0$. Uniqueness is not yet guaranteed but see below 55.4 for uniqueness in Sobolev spaces via energy methods.
Consider the Weierstrass problem

\[
\begin{align*}
U_{tt} - \alpha^2 U_{xx} &= f(t, x) & t > 0, \ x > 0 \\
U(0, x) &= U_0(x), & U_t(0, x) = U_1(x) & x > 0
\end{align*}
\]

and the reflected problem (even reflection with respect to \( x = 0 \)).

\[
\begin{align*}
\tilde{U}_{tt} - \alpha^2 \tilde{U}_{xx} &= \tilde{f}(t, x) & t > 0, \ x \in \mathbb{R} \\
\tilde{U}(0, x) &= \tilde{U}_0(x), & \tilde{U}_t(0, x) = \tilde{U}_1(x) & x \in \mathbb{R}
\end{align*}
\]

where

\[
\tilde{f}(t, x) = \begin{cases} f(t, x) & \text{if } x > 0 \\ f(t, -x) & \text{if } x < 0 \end{cases}
\]

\[
\tilde{U}_0(x) = \begin{cases} U_0(x) & \text{if } x > 0 \\ U_0(-x) & \text{if } x < 0 \end{cases}
\]
\[ u_1(x) = \begin{cases} u_1(x) & x > 0 \\ u_1(-x) & x < 0 \end{cases} \]

**Theorem 3** Problem (5) has a unique clamped solution. If \( U_0 \in C^2(\mathbb{R}), U_1 \in C^1(\mathbb{R}) \) and \( f \in C^1((0,\infty) \times \mathbb{R}) \) then the solution is given by the solution of (5').

**Proof** Homework

**Remark 4** The method of reflection can be extended to the equation on a segment with Dirichlet or Neumann boundary condition at the endpoints, see Homework.