The Heat Equation and Maximum Principle

\( \begin{cases} u_t - \alpha^2 \Delta u = f(t, x) & t > 0, \; x \in \mathbb{R}^n \\ u(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases} \) 

Equation (1)

\[ C^{1,2}(\mathbb{R}^n) = \{ g: \mathbb{R}^n \to \mathbb{R} \mid g(\cdot, x) \text{ is } C^1 \} \]

\[ \forall x \in \mathbb{R}^n, \; g(t, \cdot) \text{ is } C^2 \]

\[ \forall t \in \mathbb{R}, \; u(y) \]

Def 1. \( u \) is a classical solution of (1) if

\[ u \in C^{1,2}(0, \infty \times \mathbb{R}^n) \cap C(0, \infty \times \mathbb{R}^n) \]

Def 2. \( u \) is a weak solution of (1) if

\[ u \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^n) \text{ and } \forall \varphi \in C^0_c(\mathbb{R} \times \mathbb{R}^n) \]
\[ \int_0^\infty \int_{\mathbb{R}^n} u(x,t) \left(-\frac{\partial}{\partial t} - \alpha^2 \Delta u\right) \, dx \, dt 
\]

\[ = \int_0^\infty \int_{\mathbb{R}^n} \left[ f(x,t) \Phi(x) + \int_{\mathbb{R}^n} u_0(x) \Phi(x) \, dx \right] \, dt 
\]

\[ + \int_{\mathbb{R}^n} u_0(x) \Phi(0,x) \, dx 
\]

\[ \text{Def. 3: } u \in D'(\mathbb{R}^{n+1}) \text{ is called a generalized, distributinal solution of (1) if:} 
\]

\[ (2) \quad u_t - \alpha^2 \Delta u = f + u_0(x) \delta(t) \]

in the sense of distributions, i.e.:

\[ \left( u, -\frac{\partial}{\partial t} - \alpha^2 \Delta \Phi \right) = (f, \Phi) + (u_0(x), \Phi(0,x)) \]

\[ \forall \Phi \in D(\mathbb{R}^{n+1}). \]

Recall that the term \( u_0(x) \delta(t) \) comes either from distributinal or interpretation of

\[ \int_{\mathbb{R}^n} u_0(x) \Phi(0,x) \, dx \text{ in the weak solution} \]
As from extending classical solutions \( u \) by zero for \( t < 0 \):

\[
\tilde{u}(b, x) = \begin{cases} 
  u(b, x) & t \geq 0 \\
  0 & t < 0
\end{cases}
\]

and from differentiability preserve \( C^1 \) functions, in distributional sense, so, Lemma 10:

\[
\Rightarrow \tilde{u}_t = \begin{cases} 
  u_t & t \geq 0 \\
  0 & t < 0
\end{cases}
\]

\[
\Delta \tilde{u} = \begin{cases} 
  \Delta u & t \geq 0 \\
  0 & t < 0
\end{cases}
\]

\[
\Rightarrow \quad \tilde{u} \quad \text{satisfies}
\]

\[
\tilde{u}_t - \alpha^2 \Delta \tilde{u} = \tilde{f} + \tilde{v}_0(x) \delta(t)
\]

where \( \tilde{f}(b, x) = \begin{cases} 
  f(b, x) & t \geq 0 \\
  0 & t < 0
\end{cases} \)
Recall the fundamental solution of the heat equation: See Lecture 15.

\[ E_n(t,x) = \frac{\Theta(x)}{(4\alpha^2 t)^{n/2}} e^{-\frac{|x|^2}{4\alpha^2 t}} \]

i.e.

\[ \frac{\partial}{\partial t} E_n - \alpha^2 \Delta E_n = \delta(x) \delta(t) \]

Remark 1: (infinite propagation speed) Comparing with (2), \( E_n(t,x) \) can be interpreted as the distribution of temperature with initial data \( \nu_0(x) = \delta(x) \). The latter has compact support, while \( E_n(t,x) \neq 0 \) for all \( t > 0, x \in \mathbb{R}^n \). So the information about the nonzero temperature at \( x = 0 \) has instantly propagated in the entire space.

Recall the theorem stating that if \( f + \nu_0(x) \delta(t) \in D'(\mathbb{R}^n) \) has convolution with \( E_n \), then \( E_n \ast (f + \nu_0(x) \delta(t)) \) is the unique solution of (2) in the class of distributions that have convolution with \( E_n \), see Lecture 13.
Proposition: If \( f \in \mathcal{E}'(\mathbb{R}^{n+1}) \) (i.e. it has compact support both in time and space) and \( \psi(x) \in \mathcal{E}'(\mathbb{R}^n) \) (i.e. it has compact support), then

\[
\nu = \mathcal{E}_w * f + \mathcal{E}_w * \psi(x) \delta(t)
\]

is a solution of (2). Moreover, there are no other solutions of (2) which differ from by a distribution with compact support.

Remark 2: In general \( \nu \) above is not in \( \mathcal{E}'(\mathbb{R}^{n+1}) \). For example if \( f \equiv 0 \) and \( \psi \equiv 0 \) we get \( \nu = \mathcal{E}_w * \psi \), so \( \mathcal{E}' \) is not the appropriate class of distributions to study solutions of (2).

Let

\[
\mathcal{U} = \{ g \in \mathcal{D}'(\mathbb{R}^{n+1}) \mid \text{supp } g \subseteq \mathbb{R} \times (0,\infty) \text{ and } g \in L^\infty([0,T] \times \mathbb{R}^n) \ \forall T > 0 \}
\]

If you are not familiar with \( L^\infty \) replace it with \( g \) continuous and bounded on \( [0,T] \times \mathbb{R}^n \).
Theorem 1. If $f \in U$, then $E_u \ast f \in U$

\[ V = E_u \ast f \in U \]

Moreover, for $t > 0$,

\[ V(t,x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(z,y)}{4\pi(t-c)^2} e^{-\frac{|x-y|^2}{4(t-c)^2}} \, dx \, dy \]

\[ |V(t,x)| \leq t \sup_{(z,y) \in [0,t] \times \mathbb{R}^n} |f(z,y)| \]

\[ \lim_{t \to 0} V(t,x) = 0 \text{ uniformly for } x \in \mathbb{R}^n. \]

Hence $V$ is the unique solution in $U$ of

\[ u_t - \alpha^2 \Delta u = f; \quad u(0,x) = 0 \]

Proof. Let $\frac{h(x,y)}{2} \in D'(\mathbb{R}^n \times \mathbb{R}^n)$ approximate $1$.

\[ \lim_{t \to \infty} \left( E_u(t,x) \cdot f(z,y), \lambda \left( t, x, \frac{1}{t}, y \right) \right) = 1 \]
\[ E_u \left( t_e, x \right) \leq E_u \left( t_e, x \right) \text{ for } t \leq 0 \]

\[
\lim_{k \to \infty} \int_{\mathbb{R}^{2n}} E_u \left( k, x \right) \int_{\mathbb{R}^{2n}} f(t, y) \chi_x \left( k, x, t, y \right) \rho(k+t, k+y) \, dy \, dt \, dx \, dt.
\]

\[
x' = x + y
\]

\[
\lim_{\frac{k+1}{k} \to 0} \int_{\mathbb{R}^{2n}} \varphi(t', x') \int_{\mathbb{R}^{2n}} E_u \left( \frac{k+1}{k} - t, x'-y \right) f(t, y) \chi_x \left( \frac{k+1}{k}, x'-y, t, y \right) \, dy \, dt \, dx \, dt,
\]

Since \( \chi_x \to 1 \) and \( \| \chi_x \| \leq C \) on \( \mathbb{R}^{2n+2} \),

the Lebesgue dominated convergence theorem implies the existence of the limit (independent of \( \chi_x \) ) as long as

\[ \left| \chi_x(t', x') \right| \leq C \left( t', x' \right) f(t, y) \chi_x \left( \frac{k+1}{k}, x'-y, t, y \right) \]

which dominates the integrand is integrable on

\[ \left( t', x', t, y \right) \in \mathbb{R}^{2n+2} \]
Since \( \varphi \in C^\infty(\mathbb{R}^n) \), \( |\varphi| \leq M \) on \( \text{Supp } \varphi \) is compact, it suffices to show
\[
\mu(t', x') = \int_0^\infty \int_{\mathbb{R}^n} |E_n(t'-t, x'-y)| f(t, y) \, dy \, dx < \infty.
\]

Since \( E_n(t, x) = 0 \) for \( t < 0 \) we get:
\[
\mu(t', x') = \int_0^\infty \int_{\mathbb{R}^n} |E_n(t'-t, x'-y)| f(t, y) \, dy \, dx \leq \sup \{|f(t, y)|\} \int_0^\infty \int_{\mathbb{R}^n} |E_n(t'-t, x'-y)| \, dy \, dx \leq \frac{t'}{t} \sup \{|f(t, y)|\} \int_0^\infty \int_{\mathbb{R}^n} E_n(t, x) \, dx \, dx = \frac{t'}{t} \sup \{|f(t, y)|\}
\]
where we used \( \int_{\mathbb{R}^n} E_n(t, x) \, dx = 1 \) for \( t > 0 \).
So one can pass to the limit on page 6-7 to get
\[ (E_\nu \ast f, \Psi) = \sum_0^t \int_{\partial \Omega} \int_0^{t-\tau} E_\nu(t-\tau, x-y) f(\tau, y) \, d\sigma \, dy \, dx \, d\tau. \]

We have \(E_\nu \ast f\) is generated by the \(C^0\) function
\[ \sum_0^t \int_{\partial \Omega} \int_0^{t-\tau} E_\nu(t-\tau, x-y) f(\tau, y) \, d\sigma \, dy. \]

\[ V(t, x) = \begin{cases} 
\int_0^t \int_{\partial \Omega} \int_0^{t-\tau} E_\nu(t-\tau, x-y) f(\tau, y) \, d\sigma \, dy & t > 0 \\
0 & t \leq 0. 
\end{cases} \]

Moreover, see page 8:
\[ |V(t, x)| \leq u(t, x) \leq t \sup_{(x, y) \in [0, t] \times \partial \Omega} |f(x, y)| \]

which implies \(V \in C^0\) and also,
\[ \lim_{t \to 0} V(t, x) = 0 \quad \text{uniformly in } x \in \mathbb{R}^n. \]
Theorem 2 (Smoothness of heat potential)

If \( f \in C^{1,2}(t>0) \), \( \partial_t, \partial_x, \partial_x \partial_y, \partial_x \partial_y \partial_y \in \mathcal{W} \), then

\[
V \in C^{1,2}(t>0) \cap C([0,\infty) \times \Omega_{12^n})
\]

and \( V \) is the unique classical solution of

\[
\partial_t - \alpha^2 \nabla^2 V = f, \quad V(0,\cdot) = 0
\]

Proof. From Th 1

\[
V(t, x) = \int_0^t \int_{\Omega_{12^n}} \frac{e^{-\frac{|x-y|^2}{4\alpha^2(t-s)}} f(s, y) \, dy \, ds}{4\pi \alpha^2(t-s)^{n/2}}
\]

\[
= \frac{1}{\pi^{n/2}} \int_0^t \int_{\Omega_{12^n}} e^{-|y|^2} f(t-s, x-2\alpha \sqrt{s} y) \, dy \, ds
\]

Since \( f, f_t \in \mathcal{W} \), \( \partial V \) is continuous with the integral and

\[
\frac{\partial V}{\partial t} = \frac{1}{\pi^{n/2}} \int_{\Omega_{12^n}} e^{-|y|^2} f(0, x-2\alpha \sqrt{t} y) \, dy
\]
\[ + \frac{1}{\pi n^2} \int_0^t \int_{\mathbb{R}^n} e^{-\|y\|^2} f_x(t-x, y) \, dy \, dx \]

is continuous in \( t \) and \( x \)

\[ \frac{\partial V}{\partial t} \]

is continuous and

\[ \left| \frac{\partial V}{\partial t} (t, x) \right| \leq \sup_{y \in \mathbb{R}^n} \left| f_x(t, y) \right| + t \sup_{(2, y) \in [0, 3] \times \mathbb{R}^n} \left| f_x(2, y) \right| \]

Similarly,

\[ \frac{\partial V}{\partial x_i} (t, x) = \frac{1}{\pi n^2} \int_0^t \int_{\mathbb{R}^n} e^{-\|y\|^2} f_{x_i}(t-x, y) \, dy \, dx \\
\frac{\partial^2 V}{\partial x_i \partial x_j} (t, x) = \frac{1}{\pi n^2} \int_0^t \int_{\mathbb{R}^n} e^{-\|y\|^2} f_{x_i x_j}(t-x, y) \, dy \, dx \]

are continuous and bounded by

\[ \left| \frac{\partial V}{\partial x_i} (t, x) \right| \leq t \sup_{(2, y) \in [0, 3] \times \mathbb{R}^n} \left| f_{x_i}(2, y) \right| \]

\[ \left| \frac{\partial^2 V}{\partial x_i \partial x_j} (t, x) \right| \leq t \sup_{(2, y) \in [0, 3] \times \mathbb{R}^n} \left| f_{x_i x_j}(2, y) \right| \quad QED \]
**Theorem 3** (Single layer heat potential)

If \( u_0 \in L^\infty(\mathbb{R}^n) \) then

\[
V^{(0)} = E_\infty \ast u_0(x) \text{ for } x \in M \cap C^\infty(t > 0)
\]

\[
V^{(0)}(t, x) = \frac{\Theta(t)}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} u_0(y) e^{-\frac{|x-y|^2}{4t}} dy
\]

and

\[
|V^{(0)}(t, x)| \leq \sup_{y \in \mathbb{R}^n} |u_0(y)|.
\]

If in addition \( u_0 \in C(\mathbb{R}^n) \) then \( V^{(0)} \in C(\mathbb{R}_+ \times \mathbb{R}^n) \) and

\[
\lim_{t \to 0} V^{(0)}(t, x) = u_0(x).
\]

Hence \( V^{(0)} \) is the classical solution of

\[
Ut - \Delta u = 0, \quad u(0, x) = u_0(x).
\]

**Proof.** As in Theorem 1 let \( X_k \in D'(\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}) \) approximate unity:

\[
(\delta \times u_0 \delta, \varphi) = \lim_{k \to \infty} (E_{\infty} (t, x) \cdot u_0(y) \cdot \delta(t), X_k (t, x, y) \varphi(t+x+y)) = \lim_{k \to \infty} (E_{\infty} (t, x) \cdot u_0(y), X_k (t, x, y) \varphi(t+x+y)).
\]
\[
\lim_{x \to 0} \int_0^x P(t, x') \int_{\mathbb{R}^n} E_n(t, x'-y) v_0(y) X_n(t, x'-y, 0, y) \, dy \, dx \, dt
\]

But since \( X_n \to 1 \) and is uniformly bounded, and:

\[
h(t, x') = \int_{\mathbb{R}^n} E_n(t, x'-y) v_0(y) \, dy
\]

\[
\leq \sup_{y \in \mathbb{R}^n} (v_0(y)) \int_{\mathbb{R}^n} E_n(t, x'-y) \, dy
\]

\[
= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}
\]

dominated convergence theorem implies

\[E_n = v_0(x) \delta(t)\] is generated by the function

\[
\sqrt{(0)}(t, x) = \int_{\mathbb{R}^n} E_n(t, x-y) v_0(y) \, dy.
\]

\[
\begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}
\]

and \(|\sqrt{(0)(t, x)}| \leq h(t, x) \leq \sup_{y \in \mathbb{R}^n} (v_0(y))\)

\[
\Rightarrow \sqrt{(0)} \in \mathcal{L}
\]
For $t > 0$ we have

$$V^{(0)}(t, x) = \frac{1}{(4\pi^2 t)^{n/2}} \int_{\mathbb{R}^n} V_0(y) e^{-\frac{|x-y|^2}{4at^2}} \, dy$$

and $V^{(0)} \in C^\infty((0, \infty) \times \mathbb{R}^n)$ because the integrand is $C^\infty$ in $(t, x) \in (0, \infty) \times \mathbb{R}^n$ and all its derivatives are integrable in $y \in \mathbb{R}^n$ because they are of the form

$$u_0(y) P(x, y) e^{-\frac{|x-y|^2}{4at^2}}$$

where $u_0$ is a polynomial of degree $= \text{order of derivative}$.

Finally, if $V_0$ is continuous $\Rightarrow \lim_{t \to 0} V_0(x - 2a\sqrt{t}y) = u_0(x)$

$$\Rightarrow \lim_{t \to 0} V^0(t, x) = \lim_{t \to 0} \frac{1}{(4\pi^2 t)^{n/2}} \int_{\mathbb{R}^n} V_0(x - 2a\sqrt{t}y) e^{-\frac{|x-y|^2}{4at^2}} \, dy$$

$$= \frac{1}{(4\pi^2 t)^{n/2}} \int_{\mathbb{R}^n} V_0(x) e^{-\frac{|x-y|^2}{4at^2}} \, dy = V_0(x) QE2$$

Remark: One can actually show that any generalized (distributional) solution of

$$V_0 = a^2 \Delta u \text{ on } U \subseteq \mathbb{R}^{n+1}, \text{ \& open}$$
must be a $C^\infty$ function. This is called the
**Smoothing property of the heat equation**

**Maximum Principle** (see textbook § 3.4, page...)

Let $G \subseteq \mathbb{R}^n$ be open, bounded, and $T > 0$.

$$E_T = (0, T) \times G, \quad E_\infty = (0, \infty) \times G$$

**Theorem 4** (weak maximum principle)

If $u \in C^{1,2}(E_\infty) \cap C(E_T)$ satisfies:

$$u_t - \alpha^2 \Delta u = f(t, x) \quad \text{on } (t, x) \in E_\infty$$

and $f(t, x) \leq 0$ on $E_T$ then

$u$ attains its maximum on the bottom or lateral surface of the cylinder $E_T$, i.e.

$$\max_{(t, x) \in \overline{E_T}} u(t, x) \leq \max_{(t, x) \in \overline{E_T}} \left\{ \max_{x \in \overline{G}} u(0, x), \max_{(t, x) \in [0, T] \times \partial G} u(t, x) \right\}$$
Proof. Assume contrary. \( \exists (b_0, x_0) \in C_T : \)

\[
U(b_0, x_0) > \max \left\{ \max_{x \in \overline{C}} U(b, x), \max_{(b, x) \in [0, T] \times \partial C} U(b, x) \right\}
\]

Let \( \varepsilon = U(b_0, x_0) - M \) and

\[
U(b, x) = U(b, x) + \frac{\varepsilon - t}{2T}
\]

Then for all \( (b, x) \in \partial C \) \( U(b, x) \)

\[
U(b_0, x_0) = \varepsilon + M \geq \varepsilon + U(b, x)
\]

\[
= \varepsilon - \frac{\varepsilon}{2} + \frac{\varepsilon - t}{2T} + U(b, x) - \frac{\varepsilon}{2} + U(b, x)
\]

\[
= \varepsilon - \frac{\varepsilon}{2} + \frac{\varepsilon - t}{2T} + U(b, x)
\]

\[
\Rightarrow U \text{ attains the maximum at } (b_1, x_1) \in C_T
\]

\[
\Rightarrow \frac{\partial U}{\partial t} (b_1, x_1) \geq 0, \quad \Delta U (b_1, x_1) = 0, \quad \Delta U (b_1, x_1) \leq 0
\]

\[
\Rightarrow \frac{\partial U}{\partial t} - a^2 \Delta U \geq 0 \text{ at } (b_1, x_1) \text{ and}
\]

\[
\frac{\partial U}{\partial t} - a^2 \Delta U = \frac{\partial U}{\partial t} - \frac{\varepsilon}{2T} - a^2 \Delta U = \frac{\varepsilon}{2T} - \varepsilon < 0 \text{ on } C_T
\]

Contradiction. Q.E.D.
**Theorem 5 (Weak Maximum Principle)**

If \( u \in C^{1,2}(\Omega) \cap C(\overline{\Omega}) \) solves:

\[
    u_t - a^2 \Delta u = f(t,x) \quad \text{on} \quad (t,x) \in \Omega
\]

and \( f(t,x) \geq 0 \) on \( \Omega \), then \( u \) attains its minimum on the bottom or lateral surface of the cylinder \( \Omega_T \), i.e.

\[
    \min_{(t,x) \in \overline{\Omega}} u(t,x) \geq \min_{x \in \Gamma} \left\{ \min_{(t,x) \in [0,T] \times \partial \Omega} [u(0,x), u(t,x)] \right\}
\]

**Proof** Change \( u \) with \(-u\) in the proof of Theorem 4. Q.E.D.

**Applications of Maximum Principle:**

Theorem 6. Consider the Dirichlet boundary value problem on $G \subset \mathbb{R}^n$ bounded and open

$$\begin{cases}
\frac{\partial u}{\partial t} - \alpha^2 \Delta u = f(t,x) & t > 0, x \in G \\
u(0,x) = u_0(x) & x \in G \\
u(t,x) = g(t,x) & t > 0, x \in \partial G
\end{cases}$$

Then (3) has at most one classical solution, i.e., a solution in $C^{1/2}(\overline{G}_0) \cap C(\overline{G}_0)$

Proof. Let $u_1, u_2$ be two solutions. Then

$u = u_1 - u_2$ solves (3) with $f \equiv 0, u_0 \equiv 0, g \equiv 0$.

Theorem 4 gives $u \leq 0$ implies Theorem 5, $\Rightarrow u \geq 0$.

Hence $u \equiv 0$, Q.E.D.

2. Continuous dependence on data of solutions of Dirichlet type boundary value problems for heat equation.
Theorem 7. Consider the Dirichlet boundary value problem (3) and the related problem:

\[
\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} + \alpha^2 \Delta \tilde{u} &= f(t,x) & t > 0, x \in G \\
\tilde{u}(0,x) &= \tilde{\psi}(x) & x \in G \\
\tilde{u}(t,x) &= \tilde{g}(t,x) & \text{for } t > 0, x \in \partial G
\end{aligned}
\]

If \( T > 0 \) and \( \sup_{(t,x) \in \bar{E}_T} |f(t,x) - \tilde{f}(t,x)| \leq \delta \),

\( \sup_{x \in G} |\tilde{\psi}(x) - \psi_0(x)| \leq \epsilon_0 \),

\( \sup_{(t,x) \in [0,T] \times \partial G} |\tilde{g}(t,x) - \tilde{g}(t,x)| \leq \epsilon_1 \),

then \( \sup_{(t,x) \in \bar{E}_T} |\tilde{u}(t,x) - \tilde{u}(t,x)| \leq \max\{\epsilon_0, \epsilon_1, \delta + T\} \).

**Proof.** \( \tilde{u} = u - \tilde{u} \) solves:
\[ \begin{aligned}
\frac{\partial u}{\partial t} - \alpha^2 \nabla^2 u &= f - \tilde{f} \\
\mathbf{v}(0,x) &= u_0 - \tilde{u}_0 \\
\mathbf{v}(t,x) &= \mathbf{g} - \tilde{\mathbf{g}} - \varepsilon t \\
\end{aligned} \]

Now \( \mathbf{v}(0,x) = \mathbf{v}(t,x) - \varepsilon t \) solves

\[ \begin{aligned}
\frac{\partial \tilde{u}}{\partial t} - \alpha^2 \nabla^2 \tilde{u} &= f(t) - \tilde{f(t)} - \varepsilon x \leq 0 \quad \text{on} \quad \tilde{\mathbb{D}} \\
\tilde{u}(0,x) &= u_0 - \tilde{u}_0 \\
\tilde{u}(t,x) &= \mathbf{g} - \tilde{\mathbf{g}} - \varepsilon t \\
\end{aligned} \]

With Theorem 4 we get for \((t,x) \in \tilde{\mathbb{D}}_+

\[ \tilde{u}(t,x) \leq \max \left\{ E_0, E_1 \right\} \]

\[ \tilde{u}(t,x) \leq \max \left\{ E_0, E_1 \right\} + \varepsilon T \]

Similarly, by using \( \tilde{v}(t,x) = \mathbf{v}(t,x) - \varepsilon t \) and

Theorem 5 we get \( \tilde{v}(t,x) \geq -\max \{ E_0, E_1 \} - \varepsilon T \) Q.E.D.
Remark 3

(i) Extending Theorem 6.7 to Neumann boundary value problem, i.e. \( \partial u = g \) on \( \partial \Omega \) instead of \( u = g \), requires the strong maximum principle and Gronwall inequality, see Cloots 554.

(ii) Theorem 6.7 can be extended to unbounded domains by requiring certain behavior of the slab at infinity, for example:

\[
\lim_{|x| \to \infty} u(x,0) = 0 \quad \text{uniformly in } [0,1].
\]

See homework.

(iii) Existence of classical and generalized solutions for problem (3) will be obtained via Fourier series from properties of eigenvalues and eigenfunctions for Laplace operator, see Cloots 554.

(iv) For problem 3 on the segment or half-line (with \( \lim_{|x| \to \infty} u(x) = 0 \)) one can use the method of reflection to get one solution from the solution on the full line.