Summary

- Method of Energy for wave equation: uniqueness, continuity, dependence on data, finite propagation speed.

- Method of Reflections and applications to wave equation on half-line and on a segment.

1. Method of Energy:

Let \( \mathcal{O} \subset \mathbb{R}^n \) open, bounded and \( \partial \mathcal{O} \) piecewise \( C^1 \).

Assume \( u \in C^2((0,\infty) \times \mathcal{O}) \cap C^1((0,\infty) \times \partial \mathcal{O}) \) is a solution of:

\[
\begin{cases}
    u_{tt} - \alpha^2 \Delta u = f(t,x) & t > 0, \ x \in \mathcal{O} \\
    u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)
\end{cases}
\]

We can always find the bounded open sets \( \mathcal{O} \), with piecewise \( C^1 \) boundary such that \( \mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O} \), \( 0 < \epsilon_2 < \epsilon_1 \), and \( U \cap \mathcal{O}_2 = \emptyset \). (Why?)
Multiply the eq in (1) by $u_\varepsilon$ and integrate over $\Omega \times (0,T)$, $x \in \partial \Omega$:

$$
\int_\Omega \int_0^T \frac{1}{\varepsilon} \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right)^2 dt \, dx - a_\varepsilon \int_\Omega \int_0^T \frac{\partial u}{\partial t} \, \partial u \, dx \, dt
$$

$$
= \int_\Omega \int_0^T u_\varepsilon f(t,x) \, dx \, dt
$$

We have:

$$
\int_\Omega \int_0^T \frac{1}{\varepsilon} \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right)^2 dt \, dx = \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 (T,x) \, dx - \int_\Omega \frac{\partial u}{\partial t} (T,x) \, dx
$$

$$
\rightarrow \int_\Omega u_\varepsilon \left( \frac{\partial u}{\partial t} \right)^2 (T,x) \, dx - \int_\Omega u_\varepsilon (x) \, dx
$$

$$
\int_\Omega \int_0^T \frac{\partial u}{\partial t} \, \partial u \, dx \, dt = - \int_\Omega \int_0^T \partial \left( \nabla u \right) \cdot \nabla u \, dx \, dt
$$

$$
+ \int_\Omega \int_0^T \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} \, dx \, dt
$$

$$
\rightarrow - \frac{1}{2} \int_\Omega \left( \nabla u \right)^2 (T,x) \, dx + \frac{1}{2} \int_\Omega \left( \nabla u \right)^2 (T,x) \, dx + \int_\Omega \int_0^T \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} \, dx \, dt
$$
If

\[ \frac{\partial U}{\partial t}(t, x) = 0 \quad t > 0, \ x \in \partial \Omega \]

or

\[ \frac{\partial U}{\partial n}(t, x) = 0 \quad t > 0, \ x \in \partial \Omega \]

then the boundary term disappears. If for some \( a(x), \beta(x) \geq 0 \):

\[ a(x)U(t, x) + \beta(x) \frac{\partial U}{\partial n}(t, x) = 0 \quad t > 0, \ x \in \partial \Omega \]

then:

\[ \int_0^1 \int_{\partial \Omega} \frac{\partial U}{\partial t} \frac{\partial U}{\partial n} \, dS_x \, dt = \int_0^1 \int_{\partial \Omega} -\frac{a}{\beta} \frac{1}{2} \frac{\partial}{\partial t} [U^2] \, dS_x \, dt \]

\[ = -\frac{1}{2} \int_{\partial \Omega} \frac{d}{dt} U^2(t, x) \, dS_x + \frac{1}{2} \int_{\partial \Omega} \frac{d}{dt} V_0^2(x) \, dS_x \]

where \( \partial \Omega \) is the part of the boundary where \( \beta > 0 \).

All in all we get:

**Theorem 1 (conservation of energy)** If \( \Omega = \Omega^u \)

is bounded, open, with \( \partial \Omega \) piecewise \( C^1 \) and \( U \) is a classical solution of (1) which satisfies:

(i) \( U(t, x) = 0 \) for \( t > 0, \ x \in \partial \Omega \)

or (ii) \( \frac{\partial U}{\partial n}(t, x) = 0 \) for \( t > 0, \ x \in \partial \Omega \)

\( (\Omega^u) (x) U(t, x) + \beta(x) \frac{\partial U}{\partial n}(t, x) = 0 \) for \( t > 0, \ x \in \partial \Omega \).
Then:

\[ y(t) = y(0) + \int_0^T \int_{V_2} u_t f(t, x) \, dx \, dt. \]

Where in the case (i) and (ii):

\[ y(t) = \frac{1}{2} \int_{V_2} (\frac{\partial U}{\partial t})^2 \, dx + \frac{1}{2} \int_{V_2} (\nabla U)^2 \, dx \]

For (iii):

\[ y(t) = \frac{1}{2} \int_{V_2} (\frac{\partial U}{\partial t})^2 \, dx + \frac{\alpha^2}{2} \int_{V_2} (U^2) \, dx \]

\[ + \frac{\alpha^2}{2} \int_{S_0} \frac{\partial}{\partial n} U^2 \, dS_x. \]

Remark 1. The theorem can be obtained for more general (hyperbolic) operators, see textbook §2.4 (pages 388–390), and under less restrictive conditions on \( U \) and boundary of \( V \), see ibid., 554.
Corollary 1 (Uniqueness of BVP) For $\Omega \subset \mathbb{R}^n$ open, bounded with $\partial \Omega$ piecewise $C^1$, problem (1) has a unique classical solution satisfying

(i) $U(0, x) = g(x, x)$ if $x \geq 0, x \in \Omega$

(ii) $\frac{\partial U}{\partial n}(t, x) = g(x, x) \text{ if } t > 0, x \in \partial \Omega$

(iii) $x(x) U(x, x) + y(x) \frac{\partial U}{\partial n}(t, x) = g(x, x) \text{ if } x \geq 0, x \in \Omega$

Proof. Let $\tilde{U}_1, \tilde{U}_2$ be two such solutions, then

$U = \tilde{U}_1 - \tilde{U}_2$

satisfies (1) with $I = 0, U_0 = 0, U_1 = 0$ and zero boundary conditions. From the previous theorem we get

$J(t) = 0 \text{ if } t > 0.$

$\Rightarrow \frac{\partial U}{\partial t} = 0 \text{ and } \nabla U = 0 \text{ on } [0, T] \times \Omega$

$\Rightarrow U = C \text{ constant on } [0, T] \times \Omega \Rightarrow U = 0 \text{ since } U(0, x) = 0.$
Corollary 2. Under the hypotheses of Corollary 1, the solution of the problem depends continuously on $f$, $u_0$ and $u_1$.

Proof. See textbook § 29.2 (pages 391–394).

Corollary 3. (Domain of dependence.) Assume $u$ is a $C^2$ solution of (1) on $\mathbb{R}^n \times (0,\infty)$. Fix $(t_0,x_0) \in (0,\infty) \times \mathbb{R}^n$. If $f(t,x) \equiv 0$ in the cone $\Gamma_{(t_0,x_0)} = \{(t,x) \mid |x-x_0| < a(t-t_0)^\gamma \}$ and $u_0 = u_1 \equiv 0$ on $B(x_0,a t_0)$ then

$$u(t,x) \equiv 0 \quad \text{on} \quad (t,x) \in \Gamma_{(t_0,x_0)}$$

Proof. Define the local energy:

$$J(t) = \frac{1}{2} \int_{B(x_0,a(t-t_0))} \left( \frac{\partial u}{\partial t} \right)^2 + a^2 | \nabla u |^2 \, dx$$
\[
\frac{dy}{dt} = \int_{\Omega_{x_0, x_0+t_0-a}} \left( u_t u_x - 2u^2 + a^2 |u_x|^2 \right) dx
\]

\[
= \int_{\Omega_{x_0, x_0+t_0-a}} u_t \frac{\partial u}{\partial n} dx - \int_{\partial \Omega_{x_0, x_0+t_0-a}} 2a u_x \frac{\partial u}{\partial n} dx
\]

\[
= \int_{\Omega_{x_0, x_0+t_0-a}} u_t \frac{\partial u}{\partial n} dx - \int_{\partial \Omega_{x_0, x_0+t_0-a}} 2a u_x \frac{\partial u}{\partial n} dx
\]

But from \( |\frac{\partial u}{\partial n}| \leq |\nabla u| \) we get

\[
|2a u_x \frac{\partial u}{\partial n}| \leq u_x^2 + a^2 |\nabla u|^2
\]

\[
\frac{d}{dt} (u) \leq 0 \quad \Rightarrow \quad f(t) \leq f(0) = 0 \quad \forall 0 \leq t \leq t_0
\]

\[
\frac{d}{dt} u_x \nabla u = 0 \quad \Rightarrow \quad u = 0 \quad \text{on} \quad \Omega_{x_0, x_0+t_0-a}.
\]
Remark 2: Using Corollary 3 we can show uniqueness of classical solution of (1) for $N = N^2$. Indeed the difference of two Dirichlet solutions satisfies (1) with $f = 0$, $v_0 = v_1 = 0$. According to the Corollary the difference is zero everywhere.

Method of Reflections, see e.g. in Terras' text.

Consider the Dirichlet problem:

\[
\begin{cases}
    u_{tt} - a^2 u_{xx} = f(t, x) & t > 0, x > 0 \\
    u(0, x) = u_0(x), u_x(0, x) = u_1(x) & x > 0, \\
    u(t, 0) = 0 & t > 0
\end{cases}
\]

and the reflected problem (odd reflection with respect to $x = 0$):

\[
\begin{cases}
    \widetilde{u}_{tt} - a^2 \widetilde{u}_{xx} = \widetilde{f}(t, x) & t > 0, x \in \mathbb{R} \\
    \widetilde{u}(0, x) = \widetilde{u}_0(x), \widetilde{u}_x(0, x) = \widetilde{u}_1(x) & x \in \mathbb{R}
\end{cases}
\]
where
\[ f(t, x) = \begin{cases} \frac{1}{2} f(t, x) & \text{if } x > 0 \\ -f(t, -x) & \text{if } x < 0 \end{cases} \]

(3)
\[ \Phi_0(x) = \begin{cases} \phi_0(x) & \text{if } x > 0 \\ -\phi_0(-x) & \text{if } x < 0 \end{cases} \]

(4)
\[ \Phi_1(x) = \begin{cases} \phi_1(x) & \text{if } x > 0 \\ -\phi_1(-x) & \text{if } x < 0 \end{cases} \]

Theorem 2. Problem (2) has a unique classical solution if \( \phi_0 \in C^2(\mathbb{R}) \), \( \phi_1 \in C^1(\mathbb{R}) \) and \( f \in C^2((0, \infty) \times \mathbb{R}) \) then the solution is given by the sheet (2'), i.e.: 
\[ \Phi(t, x) = \Phi_0(t, x) \quad \forall x > 0, \quad t > 0 \]

(4) \[ \Phi(t, x) = \Phi_0(x-\alpha t) + \Phi_0(x+\alpha t) + \frac{1}{2a} \int_0^\infty \int_{x-\alpha(t-c)}^{x+\alpha(t-c)} \frac{2}{x+at} \Phi_1(y) \, dy \, dc \]

\[ + \frac{1}{2a} \int_0^\infty \int_{x-\alpha(t-c)}^{x+\alpha(t-c)} F(c, y) \, dy \, dc \quad t > 0, \ x \in \mathbb{R} \]
Proof. Uniqueness follows from the method of energy, see Remark 2.
Existence follows from Comment 2 in Lecture 16 together with

\[ V(x, 0) = \frac{V_0(-ax) + V_0(ax)}{2} + \frac{1}{2a} \int_{-a}^{a} V_1(y) \, dy \]

because $V_0$ odd

\[ -at \]

because $V_0$ odd.

\[ + \frac{1}{2a} \int_{-a(x)}^{a(x)} V_1(y) \, dy \, dx \]

because $f$ is odd in $y$

\[ = 0. \]

Remark 3. Formula (4) gives a unique solution of (2) if $f, V, V_0$ defined by (3) are locally integrable and $V_0$ is absolutely continuous (i.e. $V_0 \in L^1_{loc}(\mathbb{R})$) or, equivalently, $f, V$ are locally integrable and $V_0$ is absolutely continuous with limit $V_0(x) = 0$. Uniqueness is not yet guaranteed but see Whittaker 554 for uniqueness in Schrödinger spaces via energy methods.
Consider the Dirichlet problem
\[
\begin{align*}
\left\{ \begin{array}{l}
U_{tt} - \alpha^2 U_{xx} &= f(t, x), \quad t > 0, \quad x > 0 \\
U(0, x) &= U_0(x), \quad U_t(0, x) = U_t(x) \quad x > 0 \\
\frac{\partial U}{\partial x}(t, 0) &= 0
\end{array} \right.
\end{align*}
\]

(5)

and the reflected problem (even reflection with respect to \(x=0\)).

\[
\begin{align*}
\left\{ \begin{array}{l}
\tilde{U}_{tt} - \alpha^2 \tilde{U}_{xx} &= \tilde{f}(t, x), \quad t > 0, \quad x \in \mathbb{R}.
\end{array} \right.
\end{align*}
\]

(5')

where
\
\tilde{f}(t, x) = \left\{ \begin{array}{ll}
f(t, x) & \text{if } x > 0 \\
\tilde{f}(t, -x) & \text{if } x < 0
\end{array} \right.
\]

\(\tilde{U}_0(x) = \left\{ \begin{array}{ll}
U_0(x) & \text{if } x > 0 \\
U_0(-x) & \text{if } x < 0
\end{array} \right.\)
$$\tilde{u}_i(x) = \begin{cases} u_i(x) & x > 0 \\ u_i(-x) & x < 0 \end{cases}$$

**Theorem 3.** Problem (5) has a unique clamped solution. If $\tilde{v}_0 \in C^2(\mathbb{R}), \tilde{v}_1 \in C(\mathbb{R})$, and $\tilde{f} \in C((0,\infty) \times \mathbb{R})$ then the solution is given by the solution of (5').

**Proof**. Homework.

**Remark 4.** The method of reflection can be extended to the equation on a segment with Dirichlet or Neumann boundary condition at the endpoints, see Homework.