Introduction to Partial Diff Eq.

Def. \( u: \mathbb{R}^n \rightarrow \mathbb{R} \) unknown satisfying

\[
F(x_1, x_2, \ldots, x_n, u(x_1, x_2, \ldots, x_n), \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) = 0
\]

is called a partial diff eq.

Highest derivative appearing in (1) is the order.

If \( F(x, u_1 + u_2) = F(x, u_1) + F(x, u_2) \) \( \forall x \)

\( F(x, \lambda u_1) = \lambda F(x, u_1) \) \( \forall x \)

for any functions \( u_1, u_2 \) and any \( \lambda \in \mathbb{R} \) the eq. is called linear.

Example: Heat eq:

\[
\frac{\partial u}{\partial t}(x_1, x_2, \ldots, x_n) = k \Delta u = k \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right)
\]
Motivation

Let \( \Omega \subset \mathbb{R}^n \)

\( x_0 \in \text{Int } \Omega \)

\( B(x_0, r) \subset \Omega \)

The heat balance in \( B(x_0, r) \) can be written:

\[
Q = Q_1 + Q_2
\]

\( Q = \) heat necessary to change temperature in \( B(x_0, r) \) from \( u(t_0, x) \) to \( u(t_0 + \Delta t, x) \)

\[
Q = \int_{B} c(x) \rho(x) [u(t_0 + \Delta t, x) - u(t_0, x)] \, dx
\]

Where \( c(x) = \) specific heat, \( \rho(x) = \) density

\( Q_1 = \) heat entering \( B(x_0, r) \) through the boundary. Fourier's Law:
\[ Q_1 = \int_{t_0}^{t_0 + \Delta t} \int_{\Omega} k(x) \frac{\partial u}{\partial \mathbf{n}} (t, x) \, ds \, dt \]

\[ \text{Gauss} = \int_{t_0}^{t_0 + \Delta t} \int_{\Omega} \text{div} \left( k(x) \text{grad} u \right) \, dx \, dt \]

(Where \( k(x) \) is thermal conductivity)

\[ Q_2 = \text{Heat produced from sources inside } \Omega(x_0, x). \text{ Assume these intensity} \]
\[ i(t, x) \]

\[ Q_2 = \int_{t_0}^{t_0 + \Delta t} \int_{\Omega} i(t, x) \, dx \, dt \]

\[ Q = Q_1 + Q_2 \text{ becomes:} \]

\[ (3) \int_{\Omega} c(x) g(x) [u(t_0 + \Delta t, x) - u(t_0, x)] \, dx \]

\[ = \int_{t_0}^{t_0 + \Delta t} \int_{\Omega} \text{div} \left( k(x) \text{grad} u \right) + i(t, x) \, dx \, dt \]

Divide by \( \Delta t \) pass to the limit \( \Delta t \to 0 \).
For \( u \in C^1(\Omega, C^2(\Omega)) \) we set

\[
\int_B \left[ c(x) \frac{\partial u}{\partial t} (t_0, x) - \text{div} (k \text{grad} u) - f \right] dx = 0
\]

where \( B \subset \Omega \) arbitrary, \( t_0 \geq 0 \) arbitrary

(4) \( c(x) \frac{\partial u}{\partial t} = \text{div} (k \text{grad} u) + f \) \( \forall x \in \Omega, t \geq 0 \)

which for \( c, \varrho, k \) constants and \( f = 0 \)

is the same as (3)

Remark 1 For \( u \) nonsmooth, \( u \in C^1(\Omega, C^2(\Omega)) \)
we should solve the integral eq (3) not
the PDE (4). Weak solutions of PDE are
related to the integral form which generated them.

Remark 2 (4) or (2) have (in general)
infinitely many solutions. To be useful
in predicting temperature, initial and
boundary conditions are added:
\[ U(0, x) = V_0(x), \quad \forall x \in \Omega \] is an initial condition.

\[ U(t, x) = 0, \quad \forall x \in \partial \Omega, \ t > 0 \] is a Dirichlet boundary condition.

\[ \frac{\partial U}{\partial n} (t, x) = 0, \quad \forall x \in \partial \Omega, \ t > 0 \] is a Neumann boundary condition.

Remark 3: In practice, \( C(x), S(x), L(x), T(x), \omega(x), \Omega \) are approximately known.

Definition: A partial differential equation together with initial conditions, boundary conditions is a well-posed problem in a class of functions if:

(i) It has at least one solution in that class.

(ii) It has at most one solution in that class.

(iii) The solution depends continuously on the data (i.e. \( C, S, L, T, \omega, \Omega, \partial \Omega, \)).
Numerical Analysis
Functional Analysis
Real and Complex Analysis
Algebra
Geometry
Topology

PDE

Sciences: Physics, Chemistry, Biology, Engineering