I. First Order Eq

1. Quasilinear

\[ a_1(x_1, ..., x_n, u) \frac{\partial u}{\partial x_1} + \cdots + a_n(x_1, ..., x_n, u) \frac{\partial u}{\partial x_n} = a_n(u) \]

- **a)** Initial/boundary value problem, i.e. \( u \) is given on a hypersurface \( \Gamma_n \subseteq \mathbb{R}^n \).

  - Use Method of Characteristics if

  \[ \Gamma = \{ (x_1, ..., x_n, u(x_1, ..., x_n)) \mid (x_1, ..., x_n) \in \Gamma_n \} \]

  is not characteristic. Be able to check this condition and use the method on concrete examples, see Lecture 2 RHW 1. Note that in general the problem has infinitely many solutions if \( \Gamma \) is characteristic.

- **b)** The general solution depends on an arbitrary function and is obtained by finding \( n \) integral surfaces \( \Phi_1, ..., \Phi_n : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

  \[ \nabla \Phi_1(\nabla \Phi_1, \nabla \Phi_2, ..., \nabla \Phi_n) = u \text{ in an open set } \mathcal{U}. \]
The general solution is \( F(\Phi_1, \ldots, \Phi_n) = 0 \) where \( F : \mathbb{R} \to \mathbb{R} \) is \( C^1 \) and \( \forall F(x) \neq 0 \ \forall x \in \mathbb{R} \).

See Lecture 4 & HW 1

Remark: The solution obtained via the method of characteristics is usually local, i.e., it is defined in a neighborhood \( \mathcal{U} \) of the initial surface \( \Gamma_0 \). Such a solution may be extended to all of \( \mathbb{R}^n \) or:

- Blow-up can occur \( u \to \pm \infty \).
- Gradient catastrophe can occur, i.e., \( u \) remains bounded but \( \nabla u \to \pm \infty \). The solution may be continued past the gradient catastrophe if an appropriate weak solution has been defined, see Conservation laws.

2. Fully nonlinear:

\[
F(x, u, \nabla u) = 0
\]

a) Initial value problem - method of characteristics can be extended, see Lecture 3 & HW 1
b) General solution - method of finding integral...
surfaces does not generalize. One needs to guess (or be given) a complete integral, then the general solution can be constructed from the complete integrals are envelopes, see Lecture 4, HW 1.

3. Conservation Laws

\( \partial_t u + \partial_x F(x, u) = 0 \)

a) Def of weak solutions. Def of shock solutions, Riemann-Hugoniot condition, see Lecture 5, HW 2.

b) Entropy solution. Uniqueness of entropy solutions (Statement, no proof). Entropy conditions for shocks (Lax condition for convex/concave fluxes and its generalization for nonconvex fluxes) see Lecture 6, HW 2.

C) Entropy solutions for Riemann problems (with convex or concave fluxes), shock solutions and applications to fluid dynamics solutions of conservation laws with initial data, see Lecture 6, HW 2.
II Higher order PDE’s

\[ F(x, u, \partial^2 u) = 0 \]

1. Statement of the general Courant–Kovalevskaya theorem with initial data on a noncharacteristic hyper-surface, see Lecture 7 & HW 3

2. Classification of PDE via the number of characteristic surfaces at a point
   - elliptic
   - hyperbolic in one direction
   - parabolic

   see Lecture 8 & HW 3

3. The particular case of second order equations and second order equations with time reversible, canonical forms and applications in solving the equations, see Lecture 8 & HW 3.
III Distributions

1. Differentiation of distributions. In particular the distributional derivatives of piecewise $C^1$ or $C^2$ functions, single and double layer potentials, see Lecture 10 & HW 5.

2. Direct product and convolution of distributions, see Lecture 11 & HW 5.

3. Fourier Transform and application to fundamental solutions, see Lectures 12-16 and HW 6-7.
VI. Laplace & Poisson Eq


2. Well posedness in the entire space

3. Well posedness of the boundary value problems (no proofs for existence of solutions)

4. Green functions: def, existence & uniqueness, construction via reflections, Poisson formula and applications

See Lecture 15 new and HW 8.

IV. Heat equation

1. Well posedness in \( \mathbb{R} \), see Lecture 17.

2. Smoothness of solutions for \( t > 0 \), infinite propagation speed, see Lecture 17.
3. Weak maximum principle and applications to uniqueness and continuous dependence on data

4. Energy method in bounded domains and applications to uniqueness.

5. Method of Reflections for mixed problems

\[ \nabla \text{Wave Eq in } U = 1, 2, 3 \text{ dimensions} \]

Not required for the Final

1. Well posedness in all three classes

2. Domain of dependence, finite propagation speed.

See Lecture 18

3. Energy method and applications to finite propagation speed and uniqueness.

See Lecture 19

4. Method of reflections for mixed problems