Math 489 Midterm II
April 19, 2018

1. (a) (10 points) Let \(d, n \in \mathbb{N}\) and \(A_1, A_2, \ldots, A_n \subseteq \mathbb{R}^d\), be bounded, non-empty sets such that \(\text{boxdim}(A_j)\) exists for each \(j \in \{1, 2, \ldots, n\}\). Show that

\[
\text{boxdim}(A_1 \cup A_2 \cup \ldots \cup A_n) = \max\{\text{boxdim}(A_1), \text{boxdim}(A_2), \ldots, \text{boxdim}(A_n)\}
\]

(b) (5 points) Let \(S = \{0\} \cup \{1/2, 1/3, 1/4, \ldots\}\). Show that \(\text{boxdim}(S) = 1/2\).

(c) (5 points) Is it true that

\[
\text{boxdim} \left( \bigcup_{j \in \mathbb{N}} A_j \right) = \sup \{\text{boxdim}(A_j) \mid j \in \mathbb{N}\},
\]

where \(A_j \subseteq \mathbb{R}\) are bounded, non-empty sets for which the box dimension exists?

2. (i) (10 points) Let \(S \subseteq \mathbb{R}^d\), \(f : S \to S\) be \(C^1\) and \(p \in S\) such that \(f^k(p) = p\), for some \(k \in \mathbb{N}\). Show that the orbit of \(p\) has Lyapunov exponents given by the natural logarithm of the e-values of \(Df^k(p)\) divided by \(k\).

(ii) (10 points) Show that if the orbit of \(x\) is eventually periodic to the orbit of \(p\) then the two orbits have the same Lyapunov exponents.

3. (60 points) Consider the unit square \(\Delta = [0, 1] \times [0, 1]\) and the map \(f : \Delta \to \mathbb{R}^2\) given by

\[
f(x, y) = \begin{cases} 
(3x - \frac{1}{2}, -\frac{1}{3}y + \frac{2}{3}) & \text{for } 0 \leq x \leq \frac{1}{2}, 0 \leq y \leq 1 \\
(-4x + 4, \frac{1}{3}y) & \text{for } \frac{1}{4} \leq x \leq 1, 0 \leq y \leq 1
\end{cases}
\]

while in the middle rectangle \((1/2, 3/4) \times [0, 1]\) we have

\[
f(x, y) = (f_1(x), f_2(x, y)),
\]

which realizes a \(C^1\) smooth connection between the affine formulas in the rectangles on the left and right with \(f_1(x) > 1\) e.g. \(f_1(x) = -(4x + 1)(4x - 3)(x - \frac{1}{2}) + 1\), and \(f_2(x, y) = yg_1(x) + g_2(x)\), with \(g_{1, 2} = a_{1, 2}(8x^3 - 15x^2 + 9x) + b_{1, 2}\), where \(a_{1, 2}, b_{1, 2}\) are determined from \(g_1(1/2) = -1/3, g_1(3/4) = 1/4, \) and \(g_2(1/2) = 5/6, g_2(3/4) = 0\).

(a) (10 points) Find \(\Delta \cap f(\Delta)\) and \(\Delta \cap f^{-1}(\Delta)\).

(b) (10 points) Show that this map is a regular horseshoe map.

(c) (10 points) Find a conjugacy between the discrete dynamical system induced by this map on the invariant set:

\[
\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\Delta),
\]

and the left shift on sequences \(\{S_i\}_{i \in \mathbb{Z}}\) where \(S_i \in \{B, T\}\) for all \(i \in \mathbb{Z}\). Show that if two points \((x_1, y_1)\) and \((x_2, y_2)\) have the same itinerary:

\[
S_{-m+1} \ldots S_0 S_1 S_2 \ldots S_n
\]

then \(|x_1 - x_2| \leq (1/3)^m\) and \(|y_1 - y_2| \leq (1/3)^m\).
(d) (10 points) Explicitly find the rectangle corresponding to itinerary \( \ldots BTB \ldots \) 
Show that it has exactly one point which has a periodic orbit of period 3. Describe the orbits that asymptotically converge to this periodic orbit.

(e) (10 points) Find a chaotic orbit that starts within \((1/3)^{10} \sqrt{2}\) distance from the above periodic orbit.

(f) (10 points) Show that for this dynamical system the set of all periodic points with orbits in \( \Delta \) is countable and dense in the invariant set \( \Lambda \).
1 (a) By induction on $u \in \mathbb{N}$

For $u = 1$, $\text{Boxdim} (A_1) = \max \{ \text{Boxdim} A_i \}$

Suppose the statement is true for $u = k$. Then for $u = k+1$, we have

$$\bigcup_{j=1}^{k+1} U A_j = \left( \bigcup_{j=1}^{k} U A_j \right) \cup U A_{k+1}$$

Denote $\bigcup_{j=1}^{k} U A_j = B$. Claim:

$$\text{Boxdim} (B \cup U A_{k+1}) = \max \{ \text{Boxdim} (B), \text{Boxdim} A_{k+1} \}$$

(it suffices to prove that the algebraic core (for 2 sets) has been proven in the HW). For completeness:

Consider an $\epsilon > 0$ s.t. $\forall i$ then

$$\max \{ N_B (\epsilon), N_{A_{k+1}} (\epsilon) \} \leq N_{B \cup U A_{k+1}} (\epsilon) \leq N_B (\epsilon) + N_{A_{k+1}} (\epsilon) \leq 2 \max \{ N_B (\epsilon), N_{A_{k+1}} (\epsilon) \}$$
hence, since \( BU \) is increasing and \( BU \frac{1}{x} > 0 \) for \( x > 1 \)
we have, for \( 0 < x < 1 \):

\[
\max \left\{ \frac{\ln N_B(x)}{\ln \frac{1}{x}}, \frac{\ln N_{BU}(x)}{\ln \frac{1}{x}} \right\} \leq \frac{\ln N_{BU}(x)}{\ln \frac{1}{x}} \leq \frac{\ln 2}{\ln \frac{1}{x}} + \max \left\{ \ln N_B(x), \ln N_{BU}(x) \right\}
\]

\( \varepsilon \rightarrow 0 \)

\[ \max \{ \text{Boxclim 13}, \text{Boxclim 14a}, \text{Boxclim 14b} \} \]

By squeezed theorem \( \text{Boxclim 13} \leq \text{BU} \leq \text{Boxclim 14a}, \text{Boxclim 14b} \) and \( \varepsilon \)

\[ \max \{ \text{Boxclim 13}, \text{Boxclim 14a}, \text{Boxclim 14b} \} \]

\( \varepsilon \rightarrow 0 \)

\[ \text{Boxclim } \cup A_j \text{ exists and } \]

\[ \text{Boxclim } \bigcup_{j=1}^{k+1} A_j = \max \{ \text{Boxclim 13}, \text{Boxclim 14a}, \text{Boxclim 14b} \} \]

\( \text{inductively } \}

\[ \max \{ \text{Boxclim } A_1, \ldots, \text{Boxclim } A_k \} \]

\( \varepsilon \rightarrow 0 \)
1. (b) Use \( b_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \to 0 \). 

\[
\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \lim_{n \to \infty} \frac{1}{1 + \frac{n+n+1}{n(n+1)}} = \frac{1 + 1}{1 + 1} = 1 
\]

(use l'Hôpital for \( \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n(n+1)}} = 1 \))

\[
N(b_n) = n + 1 + \frac{u - 1}{n} = 2n - \frac{u - 1}{n} \quad \text{for } n \to \infty \quad \text{where } [0, \frac{1}{u+1}] \quad \text{all } \frac{1}{\frac{1}{n}} \quad \text{one for each } 1, \frac{1}{2}, \ldots, \frac{1}{u-1} 
\]

\[
B\text{ordini } S = \lim_{n \to \infty} \frac{\ln N(b_n)}{\ln n} = \lim_{n \to \infty} \frac{\ln 2 + \ln n}{\ln n + \ln (n+1)} = 0 + \lim_{n \to \infty} \frac{1}{1 + \frac{\ln(n+1)}{\ln n}} = 0 + \frac{1}{1 + 1} = \frac{1}{2} 
\]

1. (c) No. The example. For \( S \) in part (b)

\[
\begin{align*}
S &= \frac{\ln}{\ln \frac{1}{9}} \quad \text{Bördini } \frac{1}{9} = 0 \\
\text{Bördini } \{04\} &= 0
\end{align*}
\]
and Box dim $S = \frac{1}{2} \sup \{ 0, 0, \ldots, y = 0 \}.$

2 (a)

$$\ell (V) = \lim_{n \to \infty} \frac{\ln \| \mathbf{Df}^n (\mathbf{v}) \|_V}{n}$$

Use the subsequence \( n = kn_0 \) \( m \in \mathbb{N} \) and \( \mathbf{V} = \mathbf{V}_j \) eigenvector corresponding to 0-value \( \lambda_j \) of \( \mathbf{Df}^k (\mathbf{v}) \)

$$\ell (V_j) = \lim_{m \to \infty} \frac{\ln \| \mathbf{Df}^k (\mathbf{v})^m \mathbf{v}_j \|}{k m} = \frac{1}{k} \lim_{m \to \infty} \frac{\ln \| \mathbf{v}_j \|}{m} = \frac{\ln \| \mathbf{v}_j \|}{k}$$

Counting multiplicity of \( \lambda_j \) we have found \( m \) eigenvalues \( \nu_j \) we found all of them.

2 (ii) Assume \( f^m (x) = \mathbf{v}_j \) for some \( x \in \mathbb{N} \)

then \( \ell (x, V) = \lim_{n \to \infty} \frac{\ln \| \mathbf{Df}^n (x) \|_V}{n} = \)
\[ p(x, u) = \lim_{n \to \infty} \left( \prod_{j=1}^{n-m} f^{u-m}(x_j, u) \right) \cdot \prod_{j=1}^{n-m-1} f^{u-m-1}(x_{j+1}, u) \cdot \prod_{j=1}^{n-m-2} f^{u-m-2}(x_{j+2}, u) \cdot \cdots \right) \]|_n.

\[ = \lim_{n \to \infty} \left( \prod_{j=1}^{n-m-1} f^{u-m}(x_j, u) \right) \cdot \frac{u-m}{n}.

\[ = |\lambda_j|

for the choice of \( \psi \) such that \( Df^{u-m}(x_1) \cdots Df(x) \psi = \psi_j \).

This last part uses if a local diffeomorphism (i.e. \( Df(x), Df^{u-m} \) are all invertible).
(a) \( \Delta \cap f(\Delta) = [0, 1/3] \times [0, 1/4] \cup [0, 1/3] \times [1/2, 1/6] \)
\( \Delta \cap f^{-1}(\Delta) = [1/6, 1/2] \times [0, 1/3] \cup \left[ \frac{3}{4}, 1 \right] \times [0, 1/3] \).

Since \( f_1(x) > 1 \) for \( x \in (0, 1/6) \cup (1/2, 3/4) \).

(b) \( \Delta \cap f^{-1}(\Delta) \) are rectangles with respect to \( (x, y) \) coordinates.
\( f |_{\Delta \cap f^{-1}(\Delta)} \) is affine with a slope and direction.

(c) \( \mathcal{N} = \{ \{ S_i \}_{i \in \mathbb{N}} \mid S_i \subset [1/3, 1/4] \} \).

\[ \mathcal{N} : \lambda \rightarrow \mathcal{N} \]

\( C(x, y) = \{ S_i \}_{i \in \mathbb{N}} \) where
\[ S_i = \begin{cases} B & \text{if } f^{-1}(x, y) \in [0, 1/3] \times [0, 1/4] \\ T & \text{if } f^{-1}(x, y) \in [0, 1/3] \times [1/2, 1/6] \end{cases} \]

\[ \mathcal{C}(x) = S_{-m_1} S_{-m_2} \ldots S_0 \]

implies \( f^x(x) \in S_i \iff \exists \mathcal{C}(S_i) f^{-i}(x) = f^{-i}(S_i) \) for \( i = 0, 1, 2, \ldots, m_1 - 1 \).
Since $f$ is contracting by at least a factor of $3$, in the $y$ direction, we get:

$$|y_{1} - y_{2}| \leq (\frac{1}{3})^{n}$$

In fact:

$$|y_{1} - y_{2}| \leq \left( \frac{1}{3} \right)^{k} \left( \frac{1}{4} \right)^{n-k}$$

where $k$ counts the number of $T$ in $S_{n+1}$...

So:

$$C(x) = S_{1}S_{2}...S_{n}$$

implies:

$$f^{i}(x) \in S_{i} \iff x \in f^{-i}(S_{i}) \quad i = 1, 2, ..., n$$

since $f^{-1}$ is contracting by at least a factor of $3$ in the $x$ direction, we get:

$$|x_{1} - x_{2}| \leq \left( \frac{1}{3} \right)^{n}$$

In fact:

$$|x_{1} - x_{2}| \leq \left( \frac{1}{3} \right)^{k} \left( \frac{1}{4} \right)^{n-k}$$

where $k$ counts the number of $T$ in $S_{1}S_{2}...S_{n}$!

(a) $(x, y) \in f^{-1}(B) \cap f^{-2}(T) \cap f^{-3}(B)$

(b) $(x, y) \in [0, 1/3] \times [0, 1/4] \cap \left[ \frac{1}{2}, \frac{3}{2} \right] \times [0, 1/3]$

(c) $(x, y) \in \left[ \frac{3}{4}, 1 \right] \times [0, 1/3] \cap \left[ \frac{1}{2}, \frac{5}{6} \right]$$

$$= \left[ \frac{3}{4}, 1 \right] \times [0, \frac{3}{5}]$$
the $l^2(1, y) \in \left[ \frac{3}{4}, \frac{1}{3} \right] \cdot \left[ \frac{1}{2}, \frac{5}{6} \right]

\Rightarrow l(x, y) \in \left[ \frac{1}{2} - \frac{1}{12}, \frac{1}{2} \right] \times [0, 1, 3]

all in all $l(x, y) \in \left[ \frac{5}{12}, \frac{1}{2} \right] \times [0, 1, 3]

\Rightarrow (x, y) \in \left[ \frac{7}{8}, \frac{7}{8} + \frac{1}{4.34} \right] \times [0, 1, 3]$ is the required rectangle

So $\left[ \frac{7}{8}, \frac{7}{8} + \frac{43}{48} \right] \times [0, 1]$ is the required rectangle.

For an orbit to have period 3, the sequence BTB must repeat (uniquely). There is only one orbit.

--- BTB, BTB, BTB --- in this rectangle.

The points that have orbits asymptotically converge to ... BTB, BTB ... have the property

$C(x, y) = \text{BTB, BTB (repel)}$

which is a countable union of vertical lines.
(c) The periodic orbit has maximum length

BBTB BTBB BTB BBTB B

But the rectangle with the 20 labels determines an area, not sides smaller than \((\frac{1}{3})^{10}\) see part (c)

So the itinerary for the chaotic orbit can be

BBTB BTBB BTB BBTB BBTB BBTB BBTB BBTB BBTB B

A periodic

free to chose.

Big (c) this will not converge to a periodic orbit

and the divergence one of the Lyapunov exponents is \(\geq \ln 3\)

(f) fixed points of \(f^k\) form discrete elements

\(S_1, S_2, S_k, S_{k-1}, \ldots, S_1, S_2, S_k, S_{k-1}, \ldots, S_k\)

Hence there are \(2^k\) of them. The set of all such points is a countable union (one of \(2^k\)) of finite sets. -Count.
For clarity, fix $(x, y) \in \Lambda$ and $\epsilon > 0$. Find $m \in \mathbb{N}$ such that
\[
\left( \frac{1}{3} \right)^m \sqrt{2} < \epsilon.
\]

Then consider the periodic point
\[
(S_{m-1}, S_0, S_1, S_2, \ldots, S_m)
\]
where the last $2m$ labels coincide with the claim of $(x, y)$.